Formal Power Series and Generating Functions

In the July, 2010 Women In Mathematics video, Lauren Williams shows how to obtain the Fibonacci numbers as the coefficients of a *formal power series* or *generating function* equivalent to the expression

$$\frac{1}{1-x-x^2}.$$

Since this is a concept that even my second semester calculus students struggle with, I thought I should provide some explanation.

In imprecise terms, a *formal power series* is a polynomial of infinite degree. What do we mean by this? Let's start by describing what we mean by a *polynomial*, since this is something we may have encountered before. We'll also restrict our attention to polynomials in one variable because the one variable case already illustrates the key concepts.

A *polynomial in one variable, x,* is a mathematical expression that can be put in the form:

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$

where each $a_0, \ldots, a_n$ is a fixed number (e.g., 1, $\sqrt{2}$, or π), and x is a variable. Here are some examples of polynomials in x:

$$\begin{gathered} 1+x \\ x^2 \\ 1+2x+3x^2+4x^3+5x^4 \\ 3x^2-\tfrac{9}{4}x^5+7.3x^{100} \\ -\tfrac{5}{2}x^3+\sqrt{2}x^{10}+4.9x^{11} \end{gathered}$$

Notice that, for all of the examples above (and, indeed, for any polynomial), if we replace x by a number, then the polynomial expression also naturally represents a number. For example, in the third polynomial in the list above, if we make the substitution $x = 2$, then the polynomial expression yields the number $1 + 2(2) + 3(2)^2 + 4(2)^3 + 5(2)^4 = 129$.

Even though we can always substitute numbers for the variable x in a polynomial expression, we usually don't make a substitution when calculating with them because we rarely have to. We can calculate with polynomial expressions without having a specific value in mind for the variable x. Of course, because x represents a number, it obeys all the laws of arithmetic. Thus, for instance, $x \cdot x = x^2$ and $x + 2x = 3x$. The sum, difference, and product of two polynomials in x will also be a polynomial in x. As an example, let's explicitly compute the sum, difference, and product of $1 + x + x^2$ and $3 + 2x^2 + 5x^3$:

$$\begin{aligned} (1+x+x^2)+(3+2x^2+5x^3) &= (1+3)+(1+0)x+(1+2)x^2+(0+5)x^3 \\ &= 4+x+3x^2+5x^3, \end{aligned}$$

$$\begin{aligned} (1+x+x^2)-(3+2x^2+5x^3) &= (1-3)+(1-0)x+(1-2)x^2+(0-5)x^3 \\ &= -2+x-x^2-5x^3, \end{aligned}$$

and

$$\begin{aligned}(1+x+x^2)\cdot(3+2x^2+5x^3) &= 1\cdot(3+2x^2+5x^3)+x\cdot(3+2x^2+5x^3)+x^2\cdot(3+2x^2+5x^3)\\ &= 3+3x+5x^2+7x^3+7x^4+5x^5\end{aligned}$$

But, what about division? This is where things begin to get a little tricky. For a variety of reasons, we cannot expect that when we divide a polynomial by another polynomial, the result will be a polynomial. For example, if we divide 1 by x, the result does not have a value if we substitute $x = 0$, whereas it is possible to substitute $x = 0$ into any polynomial and get a number.

For fun, we might try to forge ahead and try to divide 1 by, say, $1 - x - x^2$, as Lauren does in the video. We can try to do this by using long division and see what happens. As Lauren shows, if we try to do this, we end up adding term after term; the long division process never terminates. We get something that looks like a polynomial, but isn't because a polynomial only has a finite number of terms. One might say that the result looks like a polynomial of "infinite degree"!

Should we reject this as nonsensical?

In fact, we don't have to reject it. We can accept it as a new mathematical object. Because it represents a sum of terms that look like a constant times a power of x, we call it a *formal power series*. The word "formal" here means that we no longer think of x as representing a number. If you try to substitute $x = 1$, for instance, into Lauren's equation, you will get

$$-1 = 1+1+2+3+5+8+13+\ldots,$$

which is absurd.

But if we think of x as a symbol that can be manipulated according to the laws of arithmetic, treating it as if it were a kind of number, then the equations make sense. We just have to remember not to substitute a particular number for x and expect to get something that makes sense. (People do study situations where substituting specific numbers in for x in a formal power series still makes sense. Understanding such substitutions is part of a subject known as *analysis*.) But as an equation involving *formal power series*, it works, and we can add, subtract, multiply, and, if the constant term is nonzero, divide by power series to get other power series.

Lauren didn't explain why long division works in this new setting (and it would take at least a decent-sized chapter in a book to explain this rigorously!), but the point is that the long division procedure is designed to ensure that the product of the denominator with the formal power series one obtains as the quotient yields the numerator.

So, a *formal power series* is like a "polynomial of infinite degree" that we manipulate *without reference to what happens when we plug in a number for the variable x*. In particular, when Lauren writes

$$\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n,$$

what she means is that if you multiply the two formal power series $1 - x - x^2$ and $\sum_{n=0}^{\infty} F_n x^n$, as follows:

$$(1-x-x^2)\cdot\left(\sum_{n=0}^{\infty} F_n x^n\right) = (1-x-x^2)\cdot(1+x+2x^2+3x^3+5x^4+\ldots)$$

you will obtain 1. I encourage you to try doing this multiplication for yourself!

I'll close by remarking that I would not be surprised if the concepts discussed in Lauren's video (and elaborated on in this not-so-short note) seem complicated and overly-abstract. This is because they are concepts that are more abstract than polynomials, and, like all abstractions, take a while to get used to. However, just like all of mathematics, the effort you put into understanding a concept will eventually pay off. Maybe you won't fully understand it this time around, but next time it will seem a bit more familiar, and eventually, after your brain percolates on it a bit, it will all click into place.

If you would like to read more about generating functions and formal power series, I suggest starting with Chapter 7 in the book *Mathematics of Choice* by Ivan Niven, followed by Chapter 8 of *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* by Miklós Bóna.