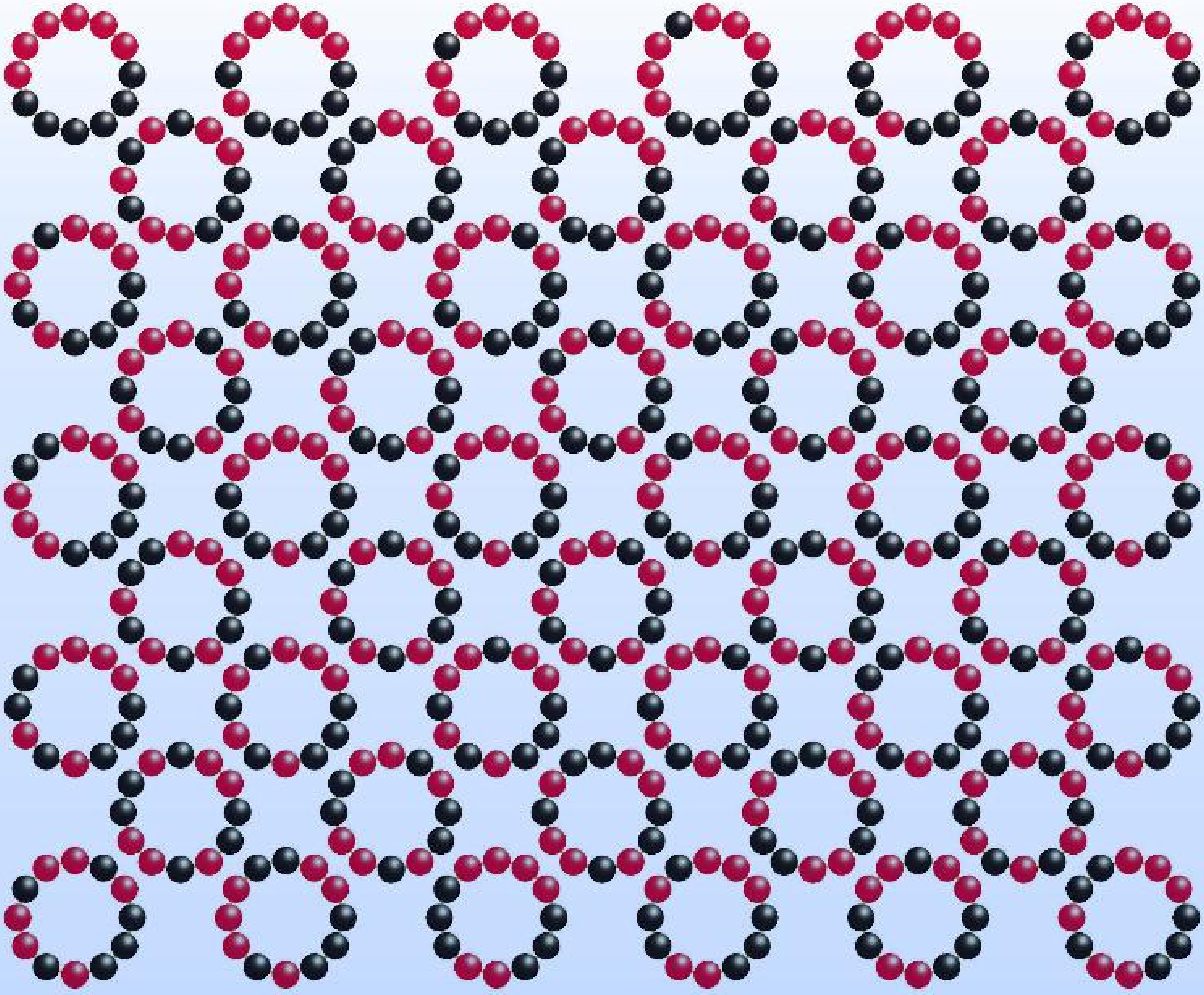
Girls' Angle Bulletin

December 2025/January 2026 • Volume 19 • Number 2

To Foster and Nurture Girls' Interest in Mathematics



An Interview with Fatima Akinola, Part 2
A Generating Function for Conjoined Compositions

Taylor Series for Tangent, Part 3
Bracelet Encoding
Notes from the Club

From the Founder

When studying math, as much as, or perhaps, even more so than the results, pay attention to the desires and questions that led to the results. Knowing the questions and desires gives all that follows a purpose. Then trust your reasoning to fill in any gaps. - Ken Fan, President and Founder

Girls' Angle Donors

A Heartfelt Thank You to Our Donors!

Individuals

Uma Achutha
Amrita Ahuja
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org.
Michael Beckman
Ravi Boppana
Dr. Emmanuouil Stylianos Brilakis
Ms. Nicole Petty Brilakis
Marla Capozzi
Suhas Daftuar
Joy and Douglas Danison
Patricia Davidson
Ingrid Daubechies
Anda Degeratu
Alexandra DeLaite and Thomas Kuo
Ellen Eischen
Glenn and Sara Ellison
Felice Frankel
Jacqueline Garrahan
Courtney Gibbons
Larry Guth
Andrea Hawksley
Scott Hilton
Delia Cheung Hom and Eugene Shih
David Kelly
Stephen Knight and Elizabeth Quattrocki Knight
Eleanor Laurans
Anna Ma
Sarah Manchester
Brian and Darlene Matthews
Toshia McCabe
Alison and Catherine Miller
Mary O'Keefe
Melina O'Grady
Beth O'Sullivan
Katherine Paur
Robert Penny and Elizabeth Tyler
Malcolm Quinn
Jeffrey and Eve Rittenberg
Marcine Snyder
Eugene Sorets
The Waldman and Romanelli Family
Brian Roughan
Patsy Wang-Iverson
Andrew Watson and Ritu Thamman
Brandy Wiegers
Lauren Williams
Brian Wilson and Annette Sassi
Lissa Winstanley
Cunyuan Zhang
The Zimmerman family
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathenaeum Foundation
Orlando Math Circle
Pro Mathematica Arte

Corporate Donors

Adobe
Big George Ventures
Google Cambridge
John Hancock
Massachusetts Innovation & Technology Exchange (MITX)
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle
Owl City Ventures, LLC

Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editors: Amanda Galtman
Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: The 50 unique bracelets consisting of 6 black beads and 6 red beads. See *Bracelet Encoding* on page 23.

An Interview with Fatima Akinola, Part 2

In general, it's good to cultivate the mindset of "ask and you shall be given." If you don't ask, no one will really tell you anything. But if you ask, people often will help you as much as they can.

This is the concluding half of our interview with University of Florida mathematics graduate student Fatima Akinola, conducted by Wellesley College undergraduate Elsa Frankel.

Elsa: Switching to your mathematics, how would you describe your field of geometric graph theory?

Fatima: I can't speak to the entire field, so I'm just going to talk about my research. My advisor and I work with a function called a "spread function."

Suppose we have a graph.[1] We imagine all the ways that we can put the nodes of the graph into the plane (i.e., draw dots in the plane, one dot for each node), say, in such a way that the distance between pairs of dots connected by an edge is at least 1. Here, we allow different nodes to be drawn in the same place (if there is no edge between the nodes, they can go to the same point). The spread function gives the minimum of the maximum distance between pairs of dots corresponding to edges over all such ways.

Let me put that into context. A very simple example is the graph consisting of the vertices and edges of a triangle. There are many ways to embed this triangle in the plane: you could realize it as an equilateral triangle or a scalene triangle. But of all the ways to put the triangle in the plane so that every edge length at least 1, the one which minimizes the maximum distance between endpoints of an edge is the equilateral triangle of side length 1. So, the spread of the triangle graph in the plane is 1.
So, what I do is try to find that spread number for general graphs. It's not as easy as it sounds because graphs can be complicated. Plus, we also look at placing graphs in other dimensions other than 2. We can also place graphs in n-dimensional Euclidean space.

This problem actually has a lot of applications, because the notion of distance itself can represent more than just physical distance. For example, you are at the airport, and think, "How do all these flights manage to not collide?" You can create a notion of distance between two flights and insist that this distance exceed some certain minimum safety limit. But it may be that it costs more to have large safety limits. And so, we can form a graph whose nodes are the flights and the distances between flights correspond to the safety limit, and we're trying to minimize the maximum safety limit between any two flights while, at the same time, requiring that the safety limit for any pair of flights exceed a certain minimum standard.

Elsa: What's an example of doing this in 3D?

Fatima: An example of looking at an embedding in $\boldsymbol{R}^3$ is the graph associated with the vertices and edges of a tetrahedron, which can be thought of as a triangle generalized to 3D. It has 4 nodes and 6

[1] Here, Fatima means a graph in the combinatorial sense, which consists of a set of nodes (or vertices) and edges, which are pairs of nodes. These are often depicted by drawing dots for the nodes and connecting pairs of nodes corresponding to edges.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on a portion of the content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

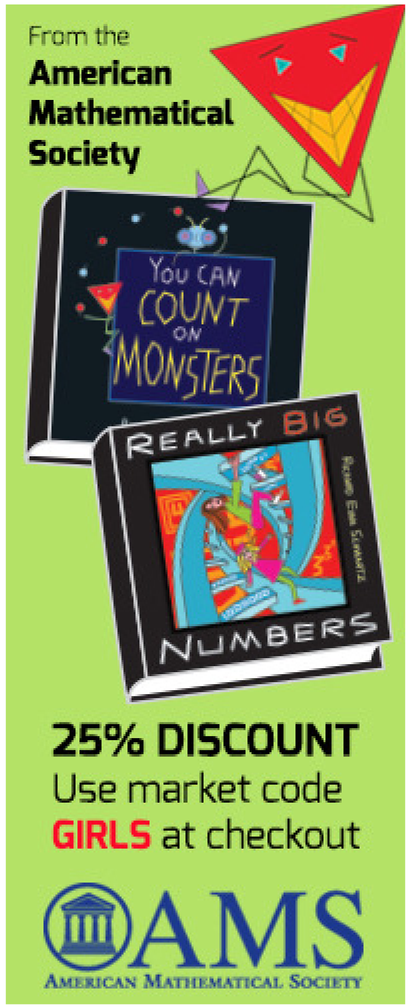
We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Content Removed from Electronic Version

Content Removed from Electronic Version



The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code “GIRLS” at checkout.

A Generating Function for Conjoined Compositions[1]

by Robert Donley[2]
edited by Amanda Galtman

In the previous installment, we developed counting formulas for semi-magic squares of size 3 and extended these results to anti-magic squares. To prove these formulas, we identified such squares with certain weak compositions subject to a relation naturally expressed with rectangles of height 2. In this part, we expand this notation to rectangles of arbitrary height. The special case of two-column rectangles leads to an interesting generating function, to which we apply some familiar techniques.

As with the previous installment, it will be helpful to recall the binomial series formula and how generating functions work ("Fibonacci Numbers and Multiset Counting," Volume 15, Number 6 and "Generating Functions for Compositions," Volume 16, Number 4).

Definition: A **weak composition** of k with p parts is an ordered list of p nonnegative integers that sum to k.

If $x_1 + . . . + x_p = k$, we can denote the composition as a p-tuple $(x_1, \ldots, x_p)$. For consistency with the rectangular notation of the previous installment, we instead use a row of boxes:

$$\begin{array}{|c|c|c|c|}\hline x_1 & x_2 & \ldots & x_p \\ \hline\end{array}$$

Recall that each anti-magic square M of size n can be identified with a weak composition with $2n$ parts in the form of a $2 \times n$ rectangle. In case several compositions represent M , all such representations are related by a shifting rule. The shifting rule is visualized below as moving a row of 1s to the other row:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 4 & 2 \\ \hline 0 & 1 & 3 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 2 & 3 & 1 \\ \hline 1 & 2 & 4 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 2 & 3 & 5 \\ \hline\end{array}$$

For counting purposes, we choose the upshifted representation, which contains at least one 0 in the lower row. Note that the sum of the entries, called the **index** of M and denoted by $i(M)$, is unchanged under the shifting rule.

Let's define the main object of interest for this installment.

Definition: A **conjoined composition** λ of k with part type (m, n) is an $m \times n$ array with nonnegative integer entries such that the entries sum to k and the rows are subject to the general shifting rule. As above, we denote the index by $i(\lambda) = k$.

This installment concerns the counting formulas for type $(m, 2)$, in which case the rectangles have two columns. In the case of part type (3, 2), the shifting rule takes the form at right, and the preferred representative, which we also call **upshifted**,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline 0 & 0 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline 1 & 1 \\ \hline\end{array}$$

[1] This installment is 24rd in a series that began in Volume 15, Number 3. This installment is the second of a new subseries.
[2] This content is supported in part by a grant from MathWorks.

has at least one zero in each of rows 2 and 3. For general m, the upshifted representative has at least one zero in all but the first row.

Let's consider a natural mathematical model encoded by such rectangles.

Example (m = 2): The case for (2, 2) represents anti-magic squares of size 2. If we revert the generating matrices to row form, then

$$R_1 = (1, 1, 0, 0), \qquad R_2 = (0, 0, 1, 1), \qquad C_1 = (1, 0, 1, 0), \qquad C_2 = (0, 1, 0, 1),$$

and all relations among these quadruples follow from

$$R_1 + R_2 = C_1 + C_2 = (1, 1, 1, 1).$$

Then $v = a_1R_1 + a_2R_2 + b_1C_1 + b_2C_2$ corresponds to the conjoined composition λ at right with index $i(\lambda) = a_1 + a_2 + b_1 + b_2$. This index is equal to half the sum of the entries of v.

a_1	a_2
b_1	b_2

Example (m = 3): Now consider the sextuples

$$A_1 = (1, 1, 1, 0, 0, 0), B_1 = (0, 1, 1, 1, 0, 0), C_1 = (0, 0, 1, 1, 1, 0),$$
$$A_2 = (0, 0, 0, 1, 1, 1), B_2 = (1, 0, 0, 0, 1, 1), C_2 = (1, 1, 0, 0, 0, 1).$$

Exercise: Prove that if $v = a_1A_1 + a_2A_2 + b_1B_1 + b_2B_2 + c_1C_1 + c_2C_2 = 0$, then

$$a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2, \text{ and } a_1 + b_1 + c_1 = 0.$$

Then prove that if two compositions represent v, they differ by the shifting rule, and that two compositions that differ by the shifting rule represent the same v.

Thus, all relations among these sextuples follow from

$$A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = (1, 1, 1, 1, 1, 1),$$

a_1	a_2
b_1	b_2
c_1	c_2

with the corresponding conjoined composition at right. The index $i(\lambda)$ equals one-third of the sum of the entries of v.

Note that the generators for the m = 3 case can be organized as the rows of the matrix at right.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise: For general m, find a set of $2m$ generators of $2m$-tuples such that

$$A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = \ldots = (1, \ldots, 1).$$

Then repeat the previous exercise for these generators.

Now let's count the number of conjoined compositions for a fixed index L. From the previous installment, we note the counting formula and generating formula for the (2, n) case.

Theorem: The number of conjoined compositions of L with part type $(2, n)$ is

$$\rho_{2,n}(L) = \binom{L+2n-1}{2n-1} - \binom{L+n-1}{2n-1}$$

with corresponding generating function

$$F_{2,n}(x) = 1 + \rho_{2,n}(1)x + \rho_{2,n}(2)x^2 + \ldots = \frac{1-x^n}{(1-x)^{2n}}.$$

For $n = 2$, we have $\rho_{2,2}(L) = (L+1)^2$ with corresponding generating function

$$F_{2,2}(x) = 1 + 4x + 9x^2 + \ldots = \frac{1-x^2}{(1-x)^4} = \frac{1+x}{(1-x)^3}.$$

Exercise: Expand the binomial coefficients in the theorem with $n = 2$ to obtain the formula for $\rho_{2,2}(L)$.

In the following exercises, we interpret the last equality for $F_{2,2}(x)$ in two ways: through binomial coefficients and through partial sums.

If we expand $\frac{1}{(1-x)^3}$ as a binomial series, then the coefficients $\binom{L+2}{2}$ are the third diagonal of Pascal's triangle: 1, 3, 6, 10, 15, 21, …. In the convolution product, the numerator of $F_{2,2}(x)$ acts as the operation of taking sums of pairs of consecutive terms, which gives the formula

$$\rho_{2,2}(L) = \binom{L+2}{2} + \binom{L+1}{2}.$$

Exercise: Verify this formula by expanding the binomial coefficients.

Exercise: Use the tabular method from the end of the previous installment to compute the first six terms of $\rho_{2,2}(L)$. That is, place a pair of 1s at the beginning of the first column, and then take three partial sum sequences.

Exercise: Describe the intermediate partial sums in the previous exercise explicitly. Then state and prove a theorem on the relationship between the second and third partial sum sequences. Also, describe the relationship between the first and second partial sum sequences.

We calculate the generating function for $\rho_{m,2}(L)$ from $F_{m-1,2}(x)$. If we remove the bottom row r of an upshifted conjoined composition λ of L of part type $(m, 2)$, we obtain a similar composition λ' of type $(m-1, 2)$. If λ' has index L, then both entries of r vanish. Otherwise, exactly one entry vanishes and the other equals some nonzero k. We obtain the formula

$$\rho_{m,2}(L) = \rho_{m-1,2}(L) + 2\rho_{m-1,2}(L-1) + 2\rho_{m-1,2}(L-2) + \ldots + 2\rho_{m-1,2}(0).$$

This formula is a product of generating functions, from which we can obtain a closed formula for $F_{m,2}(x)$. Let $P_2(x)$ be the generating function that counts the number of compositions of k with two parts, at least one of which vanishes.

Exercise: Prove that

$$P_2(x) = 1 + 2x + 2x^2 + \ldots = \frac{1+x}{1-x} = \frac{1-x^2}{(1-x)^2}.$$

Find $P_n(x)$, the generating function that counts weak compositions with n parts and at least one vanishing entry.

Since $F_{m,2}(x) = F_{m-1,2}(x)P_2(x)$, the closed-form formula follows from the formula for $F_{2,2}(x)$.

Theorem: The generating function for $\rho_{m,2}(L)$ is given by

$$F_{m,2}(x) = \frac{(1+x)^{m-1}}{(1-x)^{m+1}} = \frac{(1-x^2)^{m-1}}{(1-x)^{2m}}.$$

We calculate $\rho_{3,2}(L)$ and $\rho_{4,2}(L)$ from $F_{m,2}(x)$ in several ways. For fixed m and k with $0 \le k \le m$,

$$\binom{L-k+m}{m} = \frac{(L-k+m)\cdots(L-k+1)}{m!}$$

is a polynomial in L of degree m. By applying the binomial theorem, the binomial series, and the convolution product, we obtain the following formulas for $m = 3$ and $m = 4$, respectively:

$$\rho_{3,2}(L) = \binom{L+3}{3} + 2\binom{L+2}{4} + \binom{L+1}{4}$$

and

$$\rho_{4,2}(L) = \binom{L+4}{4} + 3\binom{L+3}{4} + 3\binom{L+2}{4} + \binom{L+1}{4}.$$

Exercise: Expand the binomial coefficients in the above formula to describe $\rho_{3,2}(L)$ as a polynomial in L.

Exercise: Describe the formula for general m. Prove that $\rho_{m,2}(L)$ is a polynomial in L of degree m and that $L + 1$ is a factor of $\rho_{m,2}(L)$.

In what follows, we consider $\rho_{m,2}(L)$ as a polynomial in L and extend the domain to all real numbers. Then the last part of the previous exercise implies that $\rho_{m,2}(-1) = 0$.

Exercise: Prove that the leading coefficient of $\rho_{m,2}(L)$ is $2^{m-1}/m!$.

Exercise: Every polynomial of degree d is determined by $d + 1$ inputs. Calculate the polynomials for $\rho_{3,2}(L)$ and $\rho_{4,2}(L)$ from the values for $L = -1, 0, 1, 2$ and the leading coefficient.

Next, we describe a recurrence for $\rho_{2,2}(L)$ by clearing the denominator in $F_{2,2}(x)$. This method works for general m, but the four term recurrence below is better for $m = 2$. We obtain

$$(1-x)^3(1+\rho_{2,2}(1)x+\rho_{2,2}(2)x^2+\ldots) = 1+x.$$

From the binomial theorem, $(1-x)^3 = 1 - 3x + 3x^2 - x^3$, so, for $L > 2$, the coefficient of x^L in the product on the left is

$$\rho_{2,2}(L) - 3\rho_{2,2}(L-1)x + 3\rho_{2,2}(L-2) - \rho_{2,2}(L-3) = 0,$$

or

$$\rho_{2,2}(L) = 3\rho_{2,2}(L-1)x - 3\rho_{2,2}(L-2) + \rho_{2,2}(L-3).$$

Exercise: Verify the recurrence for $\rho_{2,2}(L)$ by substituting $(L+1)^2$.

Exercise: Repeat the argument to find a general recurrence for $\rho_{m,2}(L)$. Where applicable, verify the recurrence for values in columns 3, 4, 5, and 6 of the right-hand table below.

Next, we analyze the generating function as a series of operations on sequences. The $P_2(x)$ product has two nice interpretations for tabular methods. To compute values for $\rho_{m,2}(L)$, we start with the (2, 2) case, the sequence of squares. The product yields the combined operation of partial sums followed by sums of consecutive pairs.

Exercise: Explain why the operations of partial sums and sums of consecutive pairs of a sequence commute; that is, explain why changing the order of operations gives the same result.

In the tables below, the one on the left shows the first two iterations of the partial sum operation (***p***) followed by the sum of consecutive pairs operation (***c***). Values of $\rho_{m,2}(L)$ are summarized in columns of the table on the right.

L	***m*=2**	***p***	***c*** ***m*=3**	***p***	***c*** ***m*=4**
0	1	1	1	1	1
1	4	5	6	7	8
2	9	14	19	26	33
3	16	30	44	70	96
4	25	55	85	155	225
5	36	91	146	301	456
6	49	140	231	532	833

L* \ *m	**2**	**3**	**4**	**5**	**6**
0	1	1	1	1	1
1	4	6	8	10	12
2	9	19	33	51	73
3	16	44	96	180	304
4	25	85	225	501	985
5	36	146	456	1182	2668
6	49	231	833	2471	6321

Exercise: Iterate the combined operation to verify the first six terms of $F_{m,2}(x)$ for $m = 5, 6$. Then search for these sequences at the Online Encyclopedia of Integer Sequences (oeis.org). Let's revisit the convolution formula above:

$$\rho_{m,2}(L) = \rho_{m-1,2}(L) + 2\rho_{m-1,2}(L-1) + 2\rho_{m-1,2}(L-2) + \ldots + 2\rho_{m-1,2}(0)$$

Exercise: Verify the entries for $\rho_{3,2}(L)$ (column 3 on the right) by choosing the same row entry in column 2 and adding twice the sum of the entries above. Then repeat for columns 4, 5, and 6.

Exercise: From the convolution formula, prove the four-term recurrence

$$\rho_{m,2}(L) = \rho_{m,2}(L-1) + \rho_{m-1,2}(L) + \rho_{m-1,2}(L-1)$$

Describe the recurrence in terms of the table entries, and use it to calculate the columns for $\rho_{m,2}(L)$ with $m = 7, 8$.

As an alternative to using binomial coefficients, we can compute formulas for $\rho_{3,2}(L)$ and $\rho_{4,2}(L)$ recursively from the convolution formula using power sum identities.

We calculate $\rho_{3,2}(L)$ as a polynomial in L from the convolution formula:

$$\begin{aligned}
\rho_{3,2}(L) &= \rho_{2,2}(L) + 2\rho_{2,2}(L-1) + 2\rho_{2,2}(L-2) + \ldots \\
&= (L+1)^2 + 2(L^2 + (L-1)^2 + \ldots) \\
&= (L+1)^2 + 2\frac{L(L+1)(2L+1)}{6} \\
&= \frac{2}{3}(L+1)^3 + \frac{1}{3}(L+1)
\end{aligned}$$

The third equality above uses the identity $1^2 + 2^2 + 3^2 + \ldots + t^2 = \dfrac{t(t+1)(2t+1)}{6}$.

Exercise: For $1 \le t \le 7$, verify the identity using the partial sum sequence for squares (the first column labeled p in the earlier table).

Exercise: Prove the sum of squares identity using induction on t. That is, if we assume the identity holds for case t, prove that it is true for case $t + 1$.

For $m = 4$, we have the polynomial formula

$$\rho_{4,2}(L) = \frac{1}{3}(L+1)(L^3 + 3L^2 + 5L + 3) = \frac{1}{3}(L+1)^4 + \frac{2}{3}(L+1)^2.$$

Exercise: For $0 \le L \le 6$, verify the polynomial formulas for $\rho_{3,2}(L)$ and $\rho_{4,2}(L)$.

Exercise: Prove the sum of cubes identity: $1^3 + 2^3 + 3^3 + \ldots + t^3 = \dfrac{t^2(t+1)^2}{4}$.

Exercise: Use this identity to derive the polynomial for $\rho_{4,2}(L)$. Then prove the second equality in the theorem using synthetic division.

The general power sum rule, given by Faulhaber's formula, depends on Bernoulli polynomials and Bernoulli numbers. For more on this formula, see the Anna's Math Journal series in the Bulletin, which runs from Volume 5, Number 1 through Volume 6, Number 1.

Next, we show a combinatorial reciprocity theorem for $\rho_{m,2}(L)$. When a counting function is a polynomial in L, replacing L with $-L$ often results in another counting function, up to a sign. In our case, we recover $\rho_{m,2}(L)$ with a shift in the variable L.

Exercise: Let $C_m(L)$ be the polynomial in L associated to the binomial coefficient $\binom{L}{m}$. Prove that $C_m(-L) = (-1)^m C_m(L + m - 1)$. What does the right-hand side count?

Since $\rho_{m,2}(L)$ is a polynomial in L, each column of the table above and on the right can be extended upward with values for $\rho_{m,2}(-L)$.

Exercise: Fill in the upper part of the table above and on the right by hand. That is, extend the second column of that table and apply the four-term recurrence to complete the column to its right. What patterns do you notice? See the end of the article for a partial table.

The main pattern in the previous exercise is summarized by the reciprocity formula

$$\rho_{m,2}(-L) = (-1)^m \rho_{m,2}(L - 2).$$

Exercise: Verify the equation directly for $m = 2, 3, 4$. What happens when $L = 1$?

The reciprocity formula enables us to refine the polynomial structure of $\rho_{m,2}(L)$. Let $P_{m,2}(L) = \rho_{m,2}(L - 1)$. Then $P_{m,2}(-L) = (-1)^m P_{m,2}(L)$, and we obtain the following theorem:

Theorem: If m is odd, then $\rho_{m,2}(L)$ is an odd polynomial of $L + 1$. If m is even, then $\rho_{m,2}(L)$ is an even polynomial of $L + 1$ with factor $(L + 1)^2$.

Exercise: Compute $\rho_{5,2}(L)$ and $\rho_{6,2}(L)$ as polynomials in $L + 1$. See if you can solve three equations in three unknowns, either by hand or with technology, to calculate $\rho_{7,2}(L)$ and $\rho_{8,2}(L)$.

Exercise: Prove the reciprocity formula by applying reciprocity for binomial coefficients to the first formula for $\rho_{4,2}(L)$. Then generalize your argument for $\rho_{m,2}(L)$.

A similar analysis of binomial coefficients describes the rows of the table for $\rho_{m,2}(L)$ as polynomials in m of degree L.

Exercise: Use the four-term recurrence to extend the table to the column for $m = -2$. Explain the column sequences for $m = 1, 0, -1$ in terms of $F_{m,2}(x)$. See the next page for a partial table.

Exercise: Find an analogue of the hockey stick rule for the four-term recurrence by expanding in the northwest direction in the fully extended table. What rules occur if we expand horizontally to the left or straight up?

Exercise: Derive formulas for $\rho_{m,2}(1)$, $\rho_{m,2}(2)$, and $\rho_{m,2}(3)$ as polynomials in m of degree k.

The preceding exercise suggests the following theorem.

Theorem: For fixed L, $\rho_{-m,2}(L) = (-1)^L\rho_{m,2}(L)$. That is, when L is odd, $\rho_{m,2}(L)$ is an odd polynomial in m, and a similar statement holds for L even.

Proof. Consider the generating function $F_{-m,2}(x)$. Then

$$F_{-m,2}(x) = \frac{(1+x)^{-m-1}}{(1-x)^{-m+1}} = \frac{(1-x)^{m-1}}{(1+x)^{m+1}} = F_{m,2}(-x).$$

If we compare the coefficients of x^L in each expansion, then the theorem follows.

Exercise: The Ehrhart polynomial $p_m(L)$ of the m-dimensional cross-polytope has generating function

$$F_m(x) = 1 + p_m(1)x + p_m(2)x^2 + \ldots = \frac{(1+x)^m}{(1-x)^{m+1}}$$

(p. 37 of the book *Counting the Continuous Discretely* by Matthias Beck and Sinai Robins). Construct the table for $p_m(L)$ by applying a single operation to the entries of the table for $\rho_{m,2}(L)$. Determine recurrences, examples of $p_m(L)$ for small m and L, and a reciprocity rule for $p_m(L)$. How does the $m = 0$ column change?

For other part types (m, n), we leave you with a final pair of exercises.

Exercise: Find a model for conjoined compositions of type (m, n) using mn-tuples.

Exercise: Find the general formula for $F_{m,n}(x)$ by imitating the recursive argument for $F_{m,2}(x)$.

Solution to the extended table of values for $\rho_{m,2}(L)$:

L \ m	-2	-1	0	1	2	3	4	5	6
-3	-4	2	0	-2	4	-6	8	-10	12
-2	1	-1	1	-1	1	-1	1	-1	1
-1	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1	1	1
1	-4	-2	0	2	4	6	8	10	12
2	9	3	1	3	9	19	33	51	73
3	-16	-4	0	4	16	44	96	180	304

Taylor Series for Tangent, Part 3

by Ken Fan | edited by Jennifer Sidney

Last time, Emily and Jasmine stumbled upon a mystery.

To recap, Emily and Jasmine have been working on finding the Taylor series for the tangent function. They started by working out the successive derivatives of $\tan(x)$, with respect to x. They found that the nth derivative of $\tan(x)$, with respect to x, has the form

$$\frac{p_n(2\sin(x))}{\cos^{n+1}(x)},$$

where $p_n(x)$ is a polynomial of degree $n - 1$ with lead coefficient 1. Furthermore, the polynomials $p_n(x)$ are defined recursively by $p_1(x) = 1$ and

$$p_{n+1}(x) = \frac{(n+1)xp_n(x) + (4 - x^2)p'_n(x)}{2},$$

for $n \geq 1$, where $p'_n(x)$ denotes the derivative of $p_n(x)$ with respect to x. Defining $c_{k,n}$ to be the coefficient of x^k in $p_n(x)$, Emily and Jasmine found that the coefficient recursion relation is

$$c_{k,n+1} = \frac{n-k+2}{2}c_{k-1,n} + 2(k+1)c_{k+1,n},$$

where $c_{-1,n}$ is defined to be 0.

Then, Emily and Jasmine noticed that if they introduced $d_{k,n} = c_{k,n}/2^{(n-k-1)/2}$,[3] then the $d_{k,n}$ will all be nonnegative integers that satisfy the recurrence relation

$$d_{k,n+1} = \frac{n-k+2}{2}d_{k-1,n} + (k+1)d_{k+1,n}.$$

They made diagrams showing the coefficients $c_{k,n}$ and $d_{k,n}$, along with numbers indicating how these coefficients can be computed inductively from the coefficients. (See the top of the next page.)

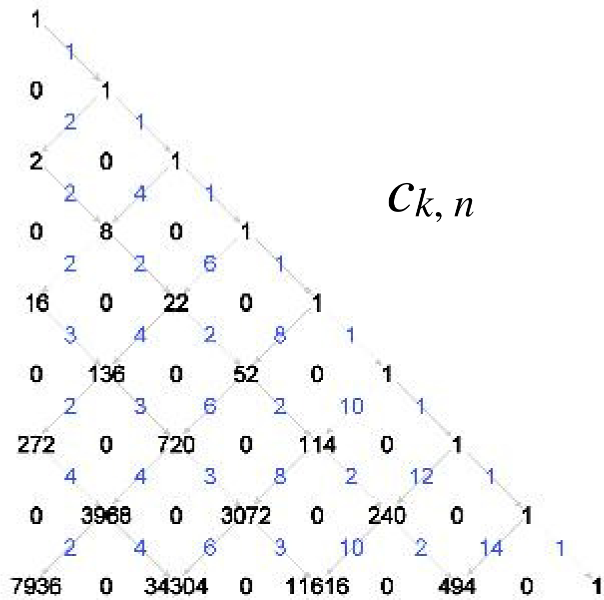
To get the Taylor series for the tangent function, the key entries are $c_{0,n} = 2^{(n-1)/2}d_{0,n}$.

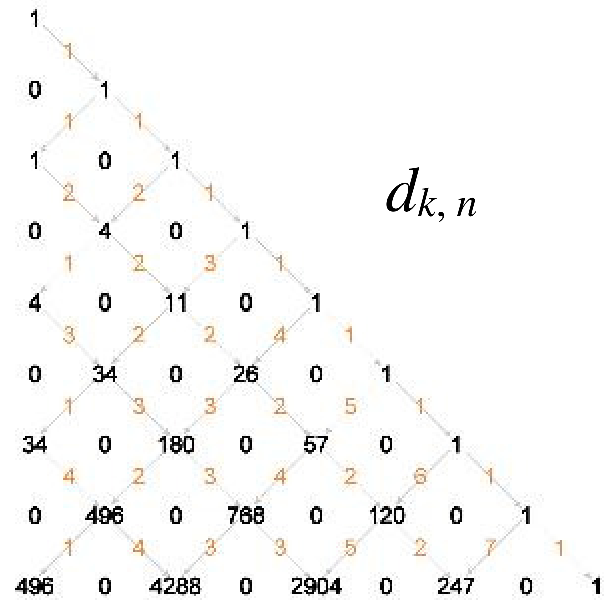
And here's where the mystery appeared:

They noticed that the sum of the $d_{k,n}$, over k with n fixed, seemed to be equal to the sequence $c_{0,n}$ when n is odd! Is this actually true? And if it is, why?

---

[3] This is a correction. In the previous installment, Emily and Jasmine had $d_{k,n} = c_{k,n}/2^{(n-k+1)/2}$, but this should have been what we have here: $d_{k,n} = c_{k,n}/2^{(n-k-1)/2}$.





The coefficients $c_{k,n}$ (left) and $d_{k,n}$ (right) arranged in triangular arrays.

Let's return to Emily and Jasmine's discussion.

Emily: The sums of the numbers in the odd-indexed rows of the $d_{k,n}$ triangle seem to give $c_{0,n}$, but I don't see a relation between the sums of the numbers in the even-indexed rows with any of the coefficients $c_{k,n}$. For the first four even-indexed rows, the sums I get are 1, 5, 61, and 1385.

Jasmine: I don't see a relation, either.

Emily: Okay, then let's just focus on trying to prove that

$$c_{0,n} = d_{0,n} + d_{1,n} + d_{2,n} + \ldots + d_{n-1,n},$$

when n is odd.

Jasmine: That's fine by me.

Emily: Maybe we can prove it by induction on n?

Jasmine: Seems worth trying…

Emily: The base case when $n = 1$ is true. So let's assume $c_{0,n} = d_{0,n} + d_{1,n} + d_{2,n} + \ldots + d_{n-1,n}$ for $n \le N$, where N is odd, and try to show that our identity is also true when $n = N + 2$.

Jasmine: Since $c_{0,n} = 2^{(n-1)/2} d_{0,n}$, I think we should eliminate $c_{0,n}$ from the identity we're trying to prove. Let's try to prove

$$2^{(n-1)/2} d_{0,n} = d_{0,n} + d_{1,n} + d_{2,n} + \ldots + d_{n-1,n}$$

instead. Or, I guess we could also express the identity using only the $c_{k,n}$ coefficients. I don't think that matters much, so let's stick with using $d_{k,n}$.

Emily: Okay.

Jasmine: So we assume that $2^{(N-1)/2}d_{0,N} = d_{0,N} + d_{1,N} + d_{2,N} + \ldots + d_{N-1,N}$ and we want to show that $2^{(N+1)/2}d_{0,N+2} = d_{0,N+2} + d_{1,N+2} + d_{2,N+2} + \ldots + d_{N+1,N+2}$, where N is odd. I guess we can use the recurrence relation for the $d_{k,n}$ to try to relate one to the other.

Emily: Using the recurrence relation,

$$d_{0,N+2} + d_{1,N+2} + d_{2,N+2} + \ldots + d_{N+1,N+2}$$

$$= d_{1,N+1} + \ldots + \frac{N+3-k}{2}d_{k-1,N+1} + (k+1)d_{k+1,N+1} + \ldots + d_{N,N+1}.$$

Jasmine: To get this in terms of coefficients in row N, we need to apply the recursion relation one more time:

$$d_{1,N+1} + \ldots + \frac{N+3-k}{2}d_{k-1,N+1} + (k+1)d_{k+1,N+1} + \ldots + d_{N,N+1}$$

$$= \frac{N+1}{2}d_{0,N} + 2d_{2,N} + \ldots + \frac{N+3-k}{2}\left(\frac{N-k+3}{2}d_{k-2,N} + kd_{k,N}\right)$$

$$+ (k+1)\left(\frac{N-k+1}{2}d_{k,N} + (k+2)d_{k+2,N}\right) + \ldots + d_{N-1,N}.$$

Emily: Gad!

Jasmine: Ugh, what a lot of algebra!

Emily: I think I'd rather try to find another way than deal with all this algebra.

Jasmine: It might not be *that* bad. We just have to stay neat and organized. I suppose we can start by combining like terms.

Emily: Hey! Instead of plowing ahead with this general approach, why don't we check this induction step for the first few cases to see if it works out cleanly. If it doesn't work nicely for the first few cases, it doesn't bode well for the general case. But if it does, then I'd be more willing to plow through all the algebra in the general case. And by doing a few of the smaller cases, maybe we'll get a clue how to organize the general computation.

Jasmine: Yes, let's do that!

Emily: So assuming that $d_{0,1} = 1$, we want to show that $d_{0,3} + d_{1,3} + d_{2,3} = 2d_{0,3}$. We have

$$d_{0,3} + d_{1,3} + d_{2,3} = d_{1,2} + 0 + d_{1,2} = 2d_{1,2} = 2d_{0,1}.$$

We want to show this is equal to $2d_{0,3}$. Actually, it would be more convenient if we combined the like terms $d_{0,3}$ in the identity we're trying to show before applying the recursion formula. I think we should try to prove that

$$0 = (1 - 2^{(N+1)/2})d_{0,N+2} + d_{1,N+2} + d_{2,N+2} + \ldots + d_{N+1,N+2},$$

for odd N. It's the same identity, but the like terms have been combined.

Jasmine: Yes, that'll save some work.

Emily: So our computation for the first induction step would go like this instead:

$$-d_{0,3} + d_{2,3} = -d_{1,2} + d_{1,2} = -d_{0,1} + d_{0,1} = 0.$$

I applied the recursion relation twice to get everything expressed in terms of the entries in the first row – though in this case we could see that the result would be 0 after applying it just once – because in general, I think we're going to have to apply the recursion relation twice.

Jasmine: Looks good! For the next step, we want to show that $-3d_{0,5} + d_{2,5} + d_{4,5} = 0$. And

$$\begin{aligned}
-3d_{0,5} + d_{2,5} + d_{4,5} &= -3d_{1,4} + (2d_{1,4} + 3d_{3,4}) + d_{3,4} \\
&= -d_{1,4} + 4d_{3,4} \\
&= -(2d_{0,3} + 2d_{2,3}) + 4d_{2,3} \\
&= -2d_{0,3} + 2d_{2,3} \\
&= 2(-d_{0,3} + d_{2,3})
\end{aligned}$$

And that is 0, since, by induction, we're assuming that $-d_{0,3} + d_{2,3} = 0$.

This is working out well so far!

Emily: Onward to the next step: showing that $-7d_{0,7} + d_{2,7} + d_{4,7} + d_{6,7} = 0$. We have

$$\begin{aligned}
-7d_{0,7} + d_{2,7} + d_{4,7} + d_{6,7} &= -7d_{1,6} + (3d_{1,6} + 3d_{3,6}) + (2d_{3,6} + 5d_{5,6}) + d_{5,6} \\
&= -4d_{1,6} + 5d_{3,6} + 6d_{5,6} \\
&= -4(3d_{0,5} + 2d_{2,5}) + 5(2d_{2,5} + 4d_{4,5}) + 6d_{4,5} \\
&= -12d_{0,5} + 2d_{2,5} + 26d_{4,5}
\end{aligned}$$

Jasmine: Oh dear. That last line does not look promising since it doesn't cleanly relate to our induction hypothesis, which, for this step, is that $-3d_{0,5} + d_{2,5} + d_{4,5} = 0$. It would have been nice if the coefficients of $d_{2,5}$ and $d_{4,5}$ were the same. Then the induction would go through easily because the final expression would have to be a multiple of $-3d_{0,5} + d_{2,5} + d_{4,5}$ (if what we're trying to prove is true).

Emily: Yeah. If we push through the algebra for the general case, we'll end up with an expression where the coefficients of the $d_{k,N}$, for $k > 0$, aren't all the same… And we'd be stuck, unable to relate it to the induction hypothesis… unless we can find other relationships between the terms in each row. Hopefully, there's another way!

Jasmine: We defined the $c_{k,n}$ as the coefficients in the polynomials $p_n(x)$, where $p_n(x)$ are used in our expression for the nth derivative of tan(x), with respect to x. Do the $d_{k,n}$ correspond to a nice sequence of polynomials?

Emily: Well, we defined $d_{k,n}$ to be $c_{k,n}/2^{(n-k-1)/2} = (\sqrt{2}^k c_{k,n}) / \sqrt{2}^{n-1}$. The number $\sqrt{2}^k c_{k,n}$ appears as the coefficient of x^k in the polynomial $p_n(\sqrt{2}x)$. So $d_{k,n}$ is the coefficient of x^k in the polynomial $p_n(\sqrt{2}x) / \sqrt{2}^{n-1}$.

Jasmine: Nice! So we're trying to prove that $2^{(n-1)/2} p_n(0) / \sqrt{2}^{n-1} = p_n(\sqrt{2}) / \sqrt{2}^{n-1}$, which can be simplified to $p_n(\sqrt{2}) = \sqrt{2}^{n-1} p_n(0)$, when n is odd.

Emily: Curious! What does it mean?

Jasmine: Let's try to relate this back to the tangent function, since I don't understand the polynomials $p_n(x)$ very well but I do have some intuition for what the tangent function is.

Emily: Great idea!

Jasmine: By our own definition, the nth derivative of tan(x), with respect to x, is

$$\frac{p_n(2\sin(x))}{\cos^{n+1}(x)}.$$

We already know that $p_n(0)$ are the numbers we need to build the Taylor series of tan(x):

$$\tan(x) = \frac{p_1(0)}{1!}x + \frac{p_3(0)}{3!}x^3 + \frac{p_5(0)}{5!}x^5 + \frac{p_7(0)}{7!}x^7 + \frac{p_9(0)}{9!}x^9 + \dots$$

So, what is $p_n(\sqrt{2})$?

Emily: If we let $x = \pi/4$, then the nth derivative of tan(x), with respect to x, evaluated at $\pi/4$ is

$$\frac{p_n(2\sin(\pi/4))}{\cos^{n+1}(\pi/4)} = \sqrt{2}^{n+1} p_n(\sqrt{2}).$$

Let's denote by $\tan^{(n)}(x)$ the nth derivative of $\tan(x)$, with respect to x. So our mystery identity that we're trying to prove is equivalent to showing that, for odd n,

$$\tan^{(n)}(\pi/4) = \sqrt{2}^{n+1} p_n(\sqrt{2}) = \sqrt{2}^{n+1}\sqrt{2}^{n-1} p_n(0) = 2^n p_n(0) = 2^n \tan^{(n)}(0).$$

Jasmine: Interesting! I've never seen that identity for derivatives of tangent before.

Emily: Neither have I. I think all the identities I know that involve the tangent function do not involve its derivatives. But, in a similar way that we could relate the $d_{k,n}$ to the polynomials $p_n(x)$, the sequence $2^n \tan^{(n)}(0)$ corresponds to the Taylor series for $\tan(2x)$.

Jasmine: Ah, good point! Replacing x with $2x$ in

$$\tan(x) = \frac{\tan^{(1)}(0)}{1!}x + \frac{\tan^{(3)}(0)}{3!}x^3 + \frac{\tan^{(5)}(0)}{5!}x^5 + \frac{\tan^{(7)}(0)}{7!}x^7 + \frac{\tan^{(9)}(0)}{9!}x^9 + \ldots$$

shows that

$$\tan(2x) = \frac{2\tan^{(1)}(0)}{1!}x + \frac{2^3\tan^{(3)}(0)}{3!}x^3 + \frac{2^5\tan^{(5)}(0)}{5!}x^5 + \frac{2^7\tan^{(7)}(0)}{7!}x^7 + \frac{2^9\tan^{(9)}(0)}{9!}x^9 + \ldots$$

Emily: But what are the numbers $\tan^{(n)}(\pi/4)$?

Jasmine: Aren't those the numbers that would be used to build the expansion of $\tan(x)$ around $x = \pi/4$ instead of $x = 0$?

Emily: You're right!

$$\tan(\pi/4 + x) = \tan(\pi/4) + \frac{\tan^{(1)}(\pi/4)}{1!}x + \frac{\tan^{(2)}(\pi/4)}{2!}x^2 + \frac{\tan^{(3)}(\pi/4)}{3!}x^3 + \ldots$$

Jasmine: Except it can't be that $\tan(\pi/4 + x) = \tan(2x)$, because the coefficients of x^k in $\tan(2x)$ are all 0 when k is even, whereas that's not the case for $\tan(\pi/4 + x)$.

Emily: That's true, but our mystery only claims that $\tan^{(n)}(\pi/4) = 2^n \tan^{(n)}(0)$ for *odd n*. If only we could magically erase the terms corresponding to x^k for even k in the expansion of $\tan(\pi/4 + x)$.

Jasmine: Wait, we can!

Emily: How?

Jasmine: We can subtract $\tan(\pi/4 - x)$ from $\tan(\pi/4 + x)$, then divide the result by 2! All the even-degree terms of $\tan(\pi/4 + x)$ will be cancelled, while all the odd-degree terms will remain as is.

Emily: Brilliant!

So our mystery claim is equivalent to showing that

$$\tan(2x) = \frac{\tan(\pi/4+x) - \tan(\pi/4-x)}{2}.$$

Now that looks more like the traditional kind of trig identity that I'm familiar with. Let's see if we can simplify the formula on the right-hand side to $\tan(2x)$. We can use the angle sum identities for $\tan(x)$ (namely, that $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$ and $\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \cdot \tan b}$) and the fact that $\tan(\pi/4) = 1$:

$$\begin{aligned}
\frac{\tan(\pi/4+x) - \tan(\pi/4-x)}{2} &= \frac{\frac{\tan(\pi/4)+\tan(x)}{1-\tan(\pi/4)\tan(x)} - \frac{\tan(\pi/4)-\tan(x)}{1+\tan(\pi/4)\tan(x)}}{2} \\
&= \frac{\frac{1+\tan(x)}{1-\tan(x)} - \frac{1-\tan(x)}{1+\tan(x)}}{2} \\
&= \frac{(1+\tan(x))^2 - (1-\tan(x))^2}{2(1-\tan(x))(1+\tan(x))} \\
&= \frac{(1+\tan(x))^2 - (1-\tan(x))^2}{2(1-\tan(x))(1+\tan(x))} \\
&= \frac{4\tan(x)}{2(1-\tan(x))(1+\tan(x))} \\
&= \frac{2\tan(x)}{1-\tan^2(x)} \\
&= \tan(2x)
\end{aligned}$$

Jasmine: It's true!

Emily: Yes!

Jasmine: I feel like celebrating at Cake Country…

Emily: Let's go!

Bracelet Encoding

by Lightning Factorial | edited by Jennifer Sidney

Do you like beaded bracelets? I do!

Not only are they an art, but they also lead to lots of math. Even in simple bracelets, there are counting problems galore. For example, here's a question inspired by a mathematics club of Nigerian girls organized by Prof. Fadipe-Joseph Olubunmi of the University of Ilorin:

There's a school with 400 students. The school would like to identify each student using a simple bracelet. Each bracelet is made out of 16 spherical beads, 8 of them black and 8 of them red. Is it possible to make 400 different bracelets so that every student can be given a unique one?

Let's try to answer this question!

In order to get a feel for the math involved, let's start with a more manageable question. Instead of making bracelets with 16 beads, let's consider bracelets with just 2 beads: 1 black and 1 red. That may seem too simple to you; but I've never thought about this before, and I like to start with the simplest example when I'm looking at something for the first time.

With 1 black and 1 red bead, we can string up the black bead first, followed by the red bead, then tie the string into a loop. Alternatively, we can string up the red bead first, followed by the black bead, then tie the string into a loop. However, it turns out that both of these scenarios actually produce the same bracelet: one can be rotated to look just like the other. So for 1 black bead and 1 red bead, only one distinct bracelet is possible.

Even though that was straightforward, I'm now mindful that we'll have to check whether two bracelets that look different are, in fact, merely rotations of each other.

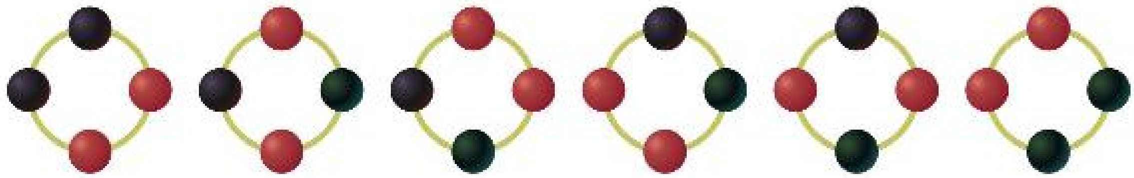
Are there any other ways two bracelets might look different but are actually the same?

I suppose we can flip bracelets, too. So we have to pull together all of the bracelet patterns that are just flips and rotations of the same pattern. By "bracelet pattern" or "pattern," I mean the sequence of bead colors as the beads are strung up before tying the ends into a loop. For 1 black and 1 red bead, there are two patterns, BR and RB, but both correspond to the same bracelet.

Let's move on to 2 black beads and 2 red beads.

There are 4 beads total, so if we imagine 4 plain beads on a bracelet, once we pick 2 of them to be black, then the other 2 must be red. Let's put aside flips and rotations for the moment, and just list all the ways 2 of the 4 beads can be chosen to be black. For the first black bead, there are 4 choices, and for the second, there are 3. However, each way of picking 2 beads to be black will be found twice, depending on which of the black beads is chosen first. Therefore, the total number of ways to pick 2 of the 4 beads to be black is $4 \times 3/2 = 6$. Let's draw them all. (See the top of the next page.)

Now let's figure out which ones are a flip and/or a rotation from each other.



It looks like the first, third, fourth, and sixth are rotations of each other, and the second and fifth are rotations of each other. That gives us two different bracelets with 2 black beads and 2 red beads.

It's interesting that the first bracelet can be rotated 90° zero, one, two, or three times, but the fourth time, it comes back to itself. By contrast, the second bracelet can only be rotated 90° zero or one time, as it looks unchanged the second time (even though same-colored beads swap positions).

We still don't have a general strategy for tackling any number of black and red beads, so let's consider the case with 3 black beads and 3 red beads.

Similar to the previous case, the total number of ways to color 3 of 6 beads black is $6 \times 5 \times 4$ divided by the number of ways each choice of 3 black beads can be ordered. Since there are 6 different orders in which the 3 black beads can be chosen, there are $6 \times 5 \times 4/6 = 20$ ways to color 3 of the 6 beads black.

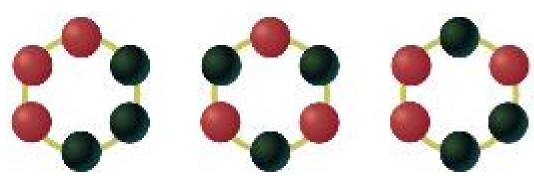
Instead of drawing the bracelets, I'll denote them by B's (for black) and R's (for red). For example, here are two bead patterns: BBBRRR and BRRBBR, where it is understood that the first and last letters represent beads that are actually next to each other in the bracelet.

Rotating the pattern BBBRRR gives six different patterns in which the black beads form a single contiguous block: BBBRRR, RBBBRR, RRBBBR, RRRBBB, BRRRBB, and BBRRRB. Flipping these over also correspond to bracelets where the black beads form a single contiguous block. So flipping and rotating yield six patterns that correspond to the same bracelet.

Rotating the pattern BBRBRR also gives six different patterns. Flipping this one, however, corresponds to the pattern RRBRBB, which represents a different bracelet pattern; none of its six rotations corresponds to a rotation of the unflipped pattern. So for BBRBRR, flipping and rotating yield 12 patterns that correspond to the same bracelet.

So far, we've accounted for 18 of the 20 patterns. What's left? It looks like BRBRBR and its rotation RBRBRB. These two patterns correspond to the same bracelet, and that accounts for all 20 patterns.

Thus, for 3 black beads and 3 red beads, we can make three different bracelets, all shown at right.



I've got an idea about how to tackle the $N = 8$ case; but if we do it in a way that is analogous to how we handled the $N = 3$ case, I think it would require a lot of work. How can we can simplify this process? To do that, we need some kind of pattern to exploit. That is, we must increase our understanding of some aspect of this problem.

So far, we have been discovering the patterns that correspond to the same bracelet by performing flips and/or rotations to the patterns and seeing what new patterns result. Let's try to better understand how many bracelet patterns give rise to the same bracelet, starting with rotations.

Suppose we have a bracelet pattern of N black beads and N red beads. To find the patterns that correspond to rotations of the same bracelet, we would rotate the bracelet by one bead again and again to see what bracelet patterns result. For definiteness, let's rotate in the clockwise direction. Since there are $2N$ beads in total, we know that after performing $2N$ such rotations, we'll return to our original pattern. But we've seen (with the pattern BRBR) that we might return to the original pattern in fewer than $2N$ rotations (by a single bead).

Let's say that M is the least positive number of single bead rotations we need to perform before we return to the original pattern. We know that $1 \leq M \leq 2N$. So if we rotate 0, 1, 2, 3, …, $M - 1$ times, we get M different patterns that all correspond to the same bracelet. When we rotate one more time, an Mth time, we're back to the original pattern and the sequence of patterns repeats through the same M patterns in the same order, before we return to the original pattern on the $2M$th rotation. If M is strictly less than $2N$, then there is a $K \geq 1$ such that K is the largest integer for which $KM < 2N$. Then $(K + 1)M \geq 2N$. Notice that $(K + 1)M - 2N < M$, because this is the inequality we get if we add M to both sides of the inequality $KM < 2N$ and then subtract $2N$ from both sides. Also, $(K + 1)M - 2N \geq 0$ (because of the way we chose K). Since $(K + 1)M - 2N$ rotations results in the same pattern as $(K + 1)M - 2N + 2N = (K + 1)M$ rotations, doing $(K + 1)M - 2N$ rotations restores the original pattern. But M is the smallest positive number of rotations we can perform to restore the original pattern! That means $(K + 1)M = 2N$. In other words, M must divide evenly into $2N$.

Also, M can never be 1 because all of our bracelets have beads of two different colors, so there must be places where a black bead and a red bead are right next to each other. If rotation by one bead results in the same pattern, then all the beads would have to be the same color.

Every bead must be the same color as the bead M beads over, since M rotations restores the pattern. This means that the first M beads in the pattern form a block that repeats $2N/M$ times to produce the entire pattern. Therefore, there must be $N/(2N/M) = M/2$ black and red beads in each block of M consecutive beads. But that means that M must be an even number. That makes me think that when N is an odd prime number, the problem will be easier because the even divisors of $2N$, when N is an odd prime, are 2 and $2N$ only. If flipping behaves well, we might be able to write down a formula for this special case. So let's turn our attention to flips.

Flipping a bracelet corresponds to reversing its bracelet pattern: BBRBRR flips to RRBRBB. So if reversing the bracelet pattern is a rotation of the pattern, then we won't get any new patterns corresponding to the same bracelet that we didn't already get by rotating. On the other hand, if reversing the bracelet pattern is not a rotation of the pattern, then we do get new patterns that correspond to the same bracelet. And since rotations of the reversed pattern have the same behavior if we just switch from clockwise to counterclockwise, we would get M new patterns (where M is the same M as in the previous paragraph).

I feel like applying this knowledge to the counting of another case. we've already worked out $N = 1$, 2, and 3. Instead of doing $N = 4$, let's try $N = 5$, since 5 is a prime number.

When $N = 5$, there are $10 \times 9 \times 8 \times 7 \times 6/5! = 252$ patterns in total. As we just determined, the value of M must be either 2 or 10. If $M = 2$, the initial block of 2 beads must contain one of each color; so the initial block must be either BR or RB, in which case the pattern is BRBRBRBRBR or RBRBRBRBRB. These are rotations of each other. Since flipping either pattern corresponds to the other, when $M = 2$, there are exactly these two patterns that give rise to the same bracelet.

All other patterns must have $M = 10$. That is, the 250 patterns with $M = 10$ can be arranged into 25 sets of 10 patterns each, where the 10 patterns in each set are rotations of each other. So the only question we have to answer is which pairs of these sets correspond to the same bracelet via a flip, and which sets "flip to themselves"?

It would be really nice if we could assume that if a set flips to itself, then there is a pattern in the set that is a palindrome. But that cannot happen because then there would have to be the same number of black beads in each half of the pattern, yet the number of black beads is odd. What must be true is that if a bracelet can be flipped then rotated so that its color pattern is the same, there must be some axis around which the bracelet is mirror symmetric. This axis will pass through the middle of two diametrically opposed beads. Furthermore, the axis must pass through a black and a red bead. We can assume, without loss of generality, that this axis is vertical, the top bead is black, and the bottom bead is red. Then 4 whole beads lie on each side of the line of symmetry. Once the color scheme for 4 beads on one side of the line of symmetry is chosen, the color pattern on the other side is determined, by mirror symmetry. On one side, 2 of the 4 beads must be black. Are all the different ways of choosing 2 of these 4 beads going to correspond to different sets of rotationally inequivalent patterns?

For the moment, I can see an ad hoc argument that all six ways of choosing the 2 black beads of the 4 (on one side of the line of symmetry) are rotationally inequivalent, except in the case where the chosen color scheme corresponds to the $M = 2$ case we found earlier. We can worry about a more general argument later. For now, let's assume that all six ways are rotationally inequivalent and complete the count. (Please don't trust me on this claim! Verify it for yourself.)

With this assumption, of the 25 sets of 10 rotationally equivalent patterns, five flip to themselves. (It's not six because one of the six mirror symmetric coloring schemes has $M = 2$, whereas these 25 sets of 10 all have $M = 10$.) The 20 remaining sets of 10 must pair up to give us 10 different bracelets.

In total, there are $10 + 5 + 1 = 16$ different bracelets that can be made with 5 black beads and 5 red beads.

Using the same reasoning for $N = 7$, we find that there are 3,432 patterns. Two of these correspond to $M = 2$, and these two are also the same bracelet after flipping. The 3,430 other patterns can be organized into 245 sets of 14 patterns that are rotationally equivalent to each other. If we create the mirror symmetric bracelet patterns, putting the line of symmetry again through a black bead at the top and a red bead at the bottom with 6 whole beads on each side of the line of symmetry, we can form $6 \times 5 \times 4/3! = 20$ patterns; one of these patterns corresponds to $M = 2$, leaving 19 of the 245 sets of 14 that "flip to themselves," with the remaining 226 pairing up to form 113 different bracelets. So for $N = 7$, there are $113 + 19 + 1 = 133$ bracelets!

Unfortunately, this still leaves the original question unanswered. And is there a general formula for the number of such bracelets as a function of N? We just have to keep thinking on it…

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 37 - Meet 9 November 6, 2025	Mentors:	Elsa Frankel, Layla Jarrahy, Kira Lewis, Yaqi Li, Sophia Liao, Hanna Mulraczyk, Maya Robinson, Dora Woodruff

Can you come up with a way of generating random positive integers so that every positive integer has a nonzero probability of occurring? Can you invent a method that makes use only of the standard ways in which we generate random numbers in real life, such as by using dice or spinners? (In this problem, try to come up with a method that does not involve having to do something infinitely many times with probability greater than 0.) After you invent a method, can you determine the probability that the positive integer N occurs?

(Note: In Volume 18, Number 5 of this Bulletin, Lai and Reinfeld looked at a random positive integer generator based on the fact that $0.\overline{9}$ equals 1. However, they did not specify a method for producing random numbers with their probability distribution. Here, we're asking about actual methods for producing random positive integers. Some of our members came up with several schemes, which led to a lot of interesting mathematics!)

Session 37 - Meet 10 November 13, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Clarise Han, Layla Jarrahy, Kira Lewis, Yaqi Li, Hanna Mularczyk, Dora Woodruff

The Chinese Remainder Theorem was a theme of the day. Perhaps this was inspired in part by Prof. Ila Fiete's Support Network visit on Meet 5, where she showed us how some animals track their location by using a system of modular counters with different moduli. The Chinese Remainder Theorem says that if p and q are relatively prime positive integers, then for any numbers x and y, with $0 \le x < p$ and $0 \le y < q$, there is a unique number z with $0 \le z < pq$ such that z leaves a remainder of x when divided by p and a remainder of y when divided by q. Even more, the correspondence respects modular arithmetic.

If you know ring theory (or group theory), the Chinese Remainder Theorem can be stated as a ring isomorphism (or group isomorphism):

$$\frac{\mathbb{Z}}{pq\mathbb{Z}} \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}},$$

where p and q are relatively prime positive integers. (The difference between considering this as a ring isomorphism versus a group isomorphism is that the group isomorphism only concerns itself with the additive structure, whereas the ring isomorphism concerns itself with both the additive and multiplicative structures.)

Session 37 - Meet 11 November 20, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Clarise Han, Layla Jarrahy, Kira Lewis, Yaqi Li, Sophia Liao, Hanna Mularczyk, Maya Robinson

The geometric series

$$a + a^2 + a^3 + \ldots + a^n$$

made a natural appearance at the club and the members before whom it appeared managed to find the identity

$$a + a^2 + a^3 + \ldots + a^n = \frac{a - a^{n+1}}{1-a}.$$

If $|a| < 1$, we can take the limit of this expression as n tends to infinity and get

$$\frac{a}{1-a} = a + a^2 + a^3 + \ldots.$$

If b is an integer greater than 1 and we set $a = 1/b$, this formula becomes

$$\frac{1}{b-1} = \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \ldots.$$

Can you see how to use this identity to quickly determine what $0.\overline{123}$ is as a fraction?

Session 37 - Meet 12 December 4, 2025	Mentors:	Elsa Frankel, Layla Jarrahy, Kira Lewis, Yaqi Li, Sophia Liao, Hanna Mularczyk, Maya Robinson, Ella Wilson

We held our traditional end-of-session Math Collaboration, designed this time by Girls' Angle mentors Yaqi Li, Hanna Mularczyk, and AnaMaria Perez. Here are some of the problems from the event:

- Given a deck of 10 cards, how many times must you "perfectly shuffle" to return back to the original order? Here, "perfectly shuffle" means to split the deck in half and interleave the cards alternating one card from one half followed by one card from the other half, and in such a way that the first card remains on top.
- A rectangular prism has its bottom and top cut off by heights 3 cm and 2 cm respectively. The resulting prism is a cube with surface area 120 cm² less than the surface area of the original rectangular prism. What was the volume of the original prism?
- Let D_8 be the dihedral group of size 8 (i.e., symmetries of a square). The 4 reflections in D_8 are split into x conjugacy classes. What is x?
- There is a six-digit number whose units digit is 6. If you move the 6 to the front, making it the first digit, the new six-digit number is 4 times the original number. What is the tens digit of the original six-digit number?

Calendar

Session 37: (all dates in 2025)

September	11	Start of the thirty-seventh session!
	19	
	25	
October	2	
	9	Ila Fiete, MIT
	16	
	23	
	30	
November	6	
	13	
	20	
	27	Thanksgiving - No meet
December	4	

Session 38: (all dates in 2026)

January	29	Start of the thirty-eighth session!
February	5	
	12	
	19	
	26	No meet
March	5	
	12	
	19	
	26	No meet
April	2	Ila Fiete, MIT
	9	
	16	
	23	No meet
	30	
May	7	

Girls' Angle has run nearly 200 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: _______________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________