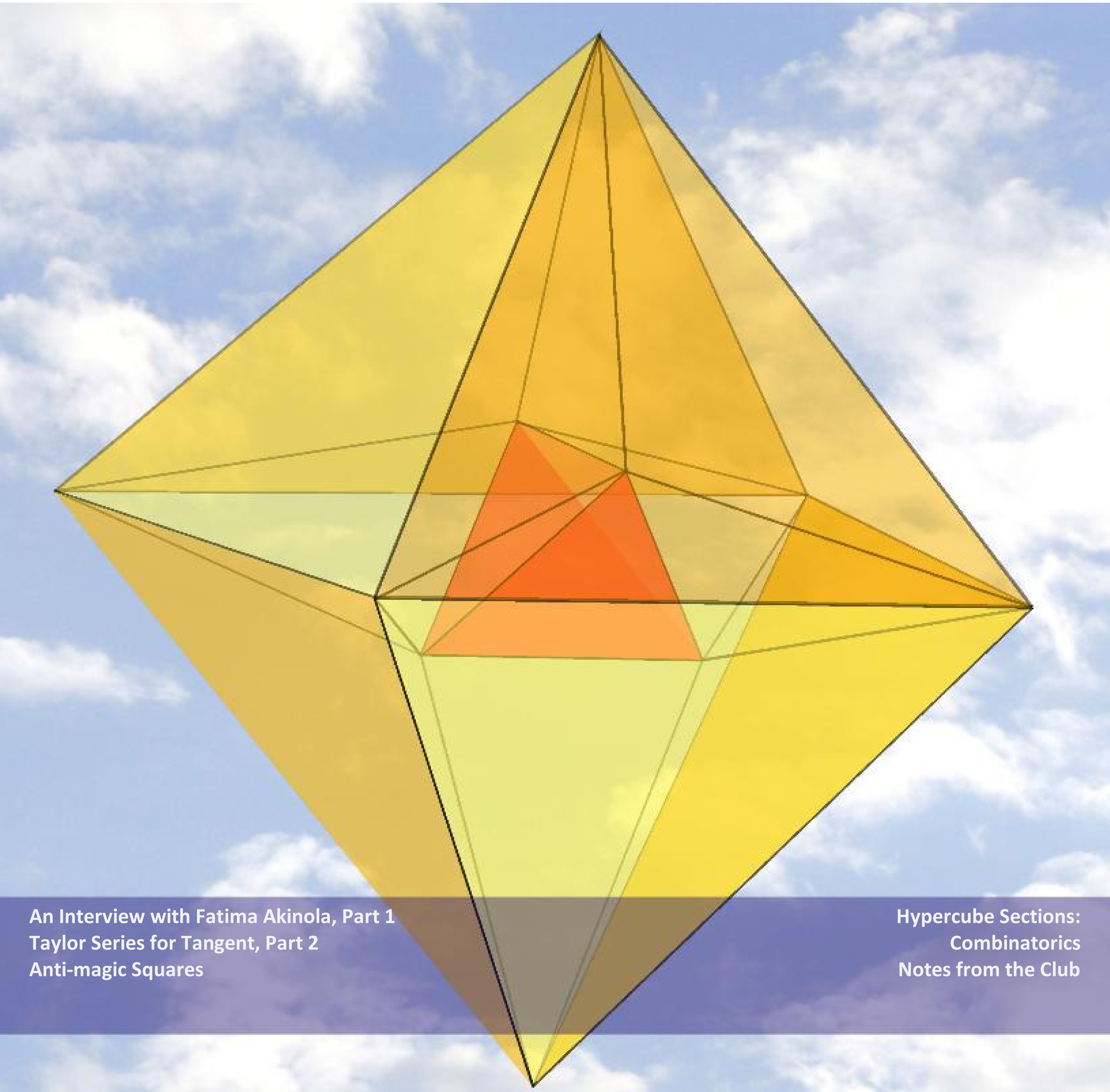
Girls' Angle Bulletin

October/November 2025 • Volume 19 • Number 1

To Foster and Nurture Girls' Interest in Mathematics



An Interview with Fatima Akinola, Part 1
Taylor Series for Tangent, Part 2
Anti-magic Squares
Hypercube Sections: Combinatorics
Notes from the Club

From the Founder

If you dislike math, you might be able to build a connection through something you do enjoy. In almost all we do that we care about, problems arise. If the problem can be thought about abstractly and logically, you've got yourself a gateway into mathematics. -Ken Fan, President and Founder

Girls' Angle Donors

A Heartfelt Thank You to our Donors!

Individuals

Uma Achutha
Amrita Ahuja
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org.
Michael Beckman
Ravi Boppana
Dr. Emmanuouil Stylianos Brilakis
Ms. Nicole Petty Brilakis
Marla Capozzi
Suhas Daftuar
Joy and Douglas Danison
Patricia Davidson
Ingrid Daubechies
Anda Degeratu
Alexandra DeLaite and Thomas Kuo
Ellen Eischen
Glenn and Sara Ellison
Felice Frankel
Jacqueline Garrahan
Courtney Gibbons
Larry Guth
Andrea Hawksley
Scott Hilton
Delia Cheung Hom and Eugene Shih
David Kelly
Stephen Knight and Elizabeth Quattrocki Knight
Eleanor Laurans
Anna Ma
Sarah Manchester
Brian and Darlene Matthews
Toshia McCabe
Alison and Catherine Miller
Mary O'Keefe
Melina O'Grady
Beth O'Sullivan
Katherine Paur
Robert Penny and Elizabeth Tyler
Malcolm Quinn
Jeffrey and Eve Rittenberg
Marcine Snyder
Eugene Sorets
The Waldman and Romanelli Family
Brian Roughan
Patsy Wang-Iverson
Andrew Watson and Ritu Thamman
Brandy Wiegers
Lauren Williams
Brian Wilson and Annette Sassi
Lissa Winstanley
Cunyuan Zhang
The Zimmerman family
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathenaeum Foundation
Orlando Math Circle
Pro Mathematica Arte

Corporate Donors

Adobe
Big George Ventures
Google Cambridge
John Hancock
Massachusetts Innovation & Technology Exchange (MITX)
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle
Owl City Ventures, LLC

Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editors: Amanda Galtman
Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: Projection of $C_{2,5}$. See *Hypercube Sections: Combinatorics* on page 20. Also known as a rectified 5-cell, it is the join of the inner tetrahedron and the outer octahedron in 4D space.

An Interview with Fatima Akinola, Part 1

Fatima Akinola is a graduate student in mathematics at the University of Florida. Her doctoral advisor is Andrew Vince. Fatima was raised in Nigeria and received her Bachelor of Science in Mathematics from Usmanu Danfodiyo University. She earned a master's degree in mathematics from Marshall University where she worked with Michael Schroeder.

This interview was conducted be Girls' Angle mentor and Wellesley College undergraduate Elsa Frankel.

Elsa: First of all, I'd like to invite you to introduce yourself and tell us how you got interested in math.

Fatima: Hello! My name is Fatima Akinola, and I am a fifth-year PhD student of mathematics at the University of Florida. I'm originally from Nigeria, and when I was in my undergrad, I wanted to study medicine, but that just did not pan out, because I wasn't given admission for it.

I was given admission to study math. I had been very good at math from a young age. I attended competitions. I represented my state in a nationwide competition. So, math just felt natural, and I didn't want to stay home, so I went ahead to study math in college.

But then I became particularly interested in math in my third and fourth years of college when I began studying real analysis, complex analysis, and numerical analysis. I was like, "Yeah, this is fun."

But my dream, actually, was to become an engineer. I thought, after studying math, I'm just going to go do a master's in engineering or something. I guess I didn't know how the world works. [Laughter] But then I graduated with a first class in math, and I pursuing a graduate degree seemed like the next best thing. I applied to schools and ended up moving to the United States.

At the time, my university was around 45 years old, and they never had a female first class student before.

Elsa: That actually answers my next question perfectly, because I was wondering how you chose to pursue graduate school. I'm curious, though, because you wanted to do engineering, but chose to get a master's degree in math at Marshall University. What led you to pursue math instead of engineering in the end?

Fatima: Graduating first class in math made pursuing math feel natural. And I did enjoy math, and it seemed that I was really good at it. I think something else that also pushed me to pursue a graduate degree in math has to do with the situation in the part of the country that I was living in. I was born and raised in northern Nigeria, and we don't get to see a lot of women in math.

Just finishing with a first class, I kind of broke history in my country. At the time, my university was around 45 years old, and they never had a female first class student before. I think that made me want to prove that a woman in math is not an alien. [Laughter] It's something that is really possible.

And so, when I pursued my graduate degree, and I came here, I did a master's at Marshall University before moving here to the University of Florida.

When I got to Marshall, of course the goal was a master's degree, but I luckily found an amazing advisor who I wanted to work with, and we did.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

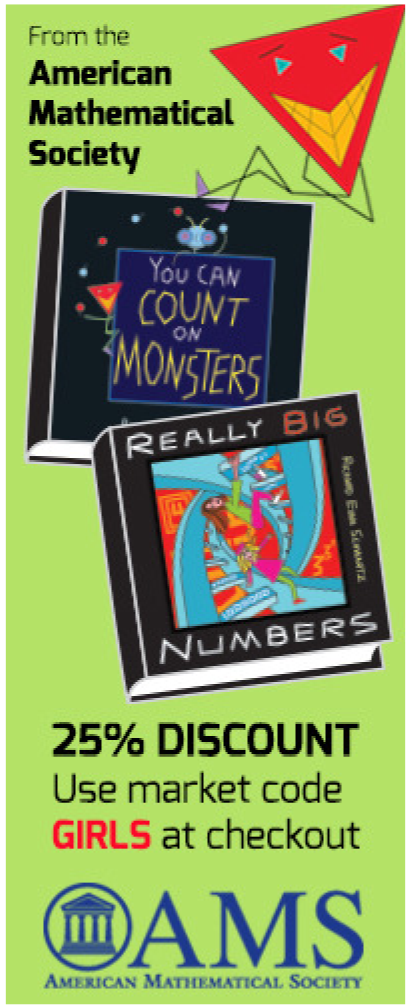
For this issue, those who do not subscribe to the print version will be missing out on a portion of the content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Content Removed from Electronic Version



The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

Content Removed from Electronic Version

Taylor Series for Tangent, Part 2

by Ken Fan | edited by Jennifer Sidney

Last time,[1] Emily and Jasmine decided to find the Taylor series expansion for $\tan(x)$. To do that, they began computing higher and higher order derivatives of $\tan(x)$, with respect to x. They found that the nth derivative of $\tan(x)$, with respect to x, has the form

$$\frac{p_n(2\sin(x))}{\cos^{n+1}(x)},$$

where $p_n(x)$ is a polynomial of degree $n-1$ with lead coefficient 1. Furthermore, the polynomials $p_n(x)$ are defined recursively by $p_1(x) = 1$ and

$$p_{n+1}(x) = \frac{(n+1)xp_n(x)+(4-x^2)p'_n(x)}{2},$$

for $n \geq 1$, where $p'_n(x)$ denotes the derivative of $p_n(x)$ with respect to x. The first few polynomials $p_n(x)$ are

n	$p_n(x)$
1	1
2	x
3	$2 + x^2$
4	$8x + x^3$
5	$16 + 22x^2 + x^4$
6	$136x + 52x^3 + x^5$
7	$272 + 720x^2 + 114x^4 + x^6$
8	$3968x + 3072x^3 + 240x^5 + x^7$
9	$7936 + 34304x^2 + 11616x^4 + 494x^6 + x^8$

Let's rejoin Emily and Jasmine's conversation.

Emily: What else can we say about these polynomials?

Jasmine: It looks like the polynomials alternate between being even and odd functions because the powers of x alternate between being all even or all odd.

Emily: Yes, that's reflected in the recurrence relation. If $p_n(x)$ is an odd function, then $xp_n(x)$ will only have even powers of x, and $(4 - x^2)p'_n(x)$ will also only have even powers of x since $p'_n(x)$ will only have even powers of x. A similar argument applies if $p_n(x)$ is an even function.

[1] See Volume 18, Number 4 of this *Bulletin*.

Jasmine: That proves it! Actually, I think this also follows from $\tan(x)$ being an odd function.

Emily: Oh, I see what you're saying. The derivative of an odd function is even, and the derivative of an even function is odd, so $\dfrac{p_n(2\sin(x))}{\cos^{n+1}(x)}$ will alternate between odd and even.

Jasmine: Exactly. And $\cos(x)$ is even, so $\dfrac{p_n(2\sin(x))}{\cos^{n+1}(x)}$ is even if and only if $p_n(2\sin(x))$ is even, and it's odd if and only if $p_n(2\sin(x))$ is odd. And since $2\sin(x)$ is odd, $p_n(2\sin(x))$ is even if and only if $p_n(x)$ is even, and it's odd if and only if $p_n(x)$ is odd.

Emily: I love it when you can see something conceptually!

Jasmine: Do you think all of these polynomials have integer coefficients?

Emily: It does seem that way. Well, in the recurrence relation, the computation of the numerator only involves operations that will result in integer values, so all we have to check is that the numerator is a polynomial whose coefficients are all even.

Jasmine: Okay. So suppose that $p_n(x) = c_0 + c_1x + c_2x^2 + \ldots + c_{n-1}x^{n-1}$. We know that either the even or the odd coefficients will all be zero, but we don't know which, so I'm just going to leave them all as variables.

Emily: Okay.

Jasmine: Then $p'_n(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots + (n-1)c_{n-1}x^{n-2}$.

Emily: Yes.

Jasmine: The coefficient of x^k in the numerator, $(n + 1)xp_n(x) + (4 - x^2)p'_n(x)$, would then be

$$(n + 1)c_{k-1} + 4(k + 1)c_{k+1} - (k - 1)c_{k-1},$$

which can be simplified to

$$(n - k + 2)c_{k-1} + 4(k + 1)c_{k+1}.$$

Emily: The second term, $4(k + 1)c_{k+1}$, will always be even, so we have to show that the first term is always even.

Jasmine: It doesn't look like it has to be even…

Emily: Maybe this is where we have to use the even/odd nature of these polynomials. If n is odd, then $p_n(x)$ is an even function, so $c_k = 0$ for all odd k. Therefore, $(n - k + 2)c_{k-1}$ is nonzero only when k is odd, and if k is odd, then $n - k + 2$ is even. So that works!

Jasmine: Nice! And if n is even, then $p_n(x)$ is an odd function, so $c_k = 0$ for all even k; this means that $(n - k + 2)c_{k-1}$ is nonzero only when k is even, and if k is even, then $n - k + 2$ is also even! The polynomials do have integer coefficients!

Emily: Since $k \leq n$, our formula for the coefficients also shows that the coefficient of x^k in $p_k(x)$ is, in fact, positive for odd k when p_n is odd, or for even k when p_n is even.

Jasmine: I think that means that when $p_n(x)$ is an even function, it has no real roots, and if it is an odd function, then its only real root is $x = 0$. This follows from symmetry, and also because as soon as $x > 0$, then $p_n(x) > 0$.

Emily: In fact, if we fix k, then $c_{k,n}$ strictly increases for $k \geq n - 1$, so $p_{n+2}(x) > p_n(x)$ for all $x > 0$. These properties of $p_n(x)$ are amusing; but to compute the Taylor series for $\tan(x)$, we only need the values of the derivatives of $\tan(x)$ at $x = 0$, which means we only need to know $p_n(0)$.

Jasmine: Good point. We know $p_n(0) = 0$ for even n, so what is $p_n(0)$ for odd n?

Emily: Oh dear.

Jasmine: What's the matter?

Emily: The recursion formula involves the first derivative of $p_n(x)$, with respect to x. That means that the constant term of $p_n(x)$ will have contributions from the linear term of $p_{n-1}(x)$, which will have contributions from the quadratic term of $p_{n-2}(x)$, etc. So it seems that to compute $p_n(0)$ with our recurrence relation, we still have to compute all the coefficients of $p_n(x)$.

Jasmine: No wonder they don't derive the Taylor series of $\tan(x)$ in school!

Emily: Let's let $c_{k,n}$ be the coefficient of x^k in $p_n(x)$. Then our coefficient recursion relation is

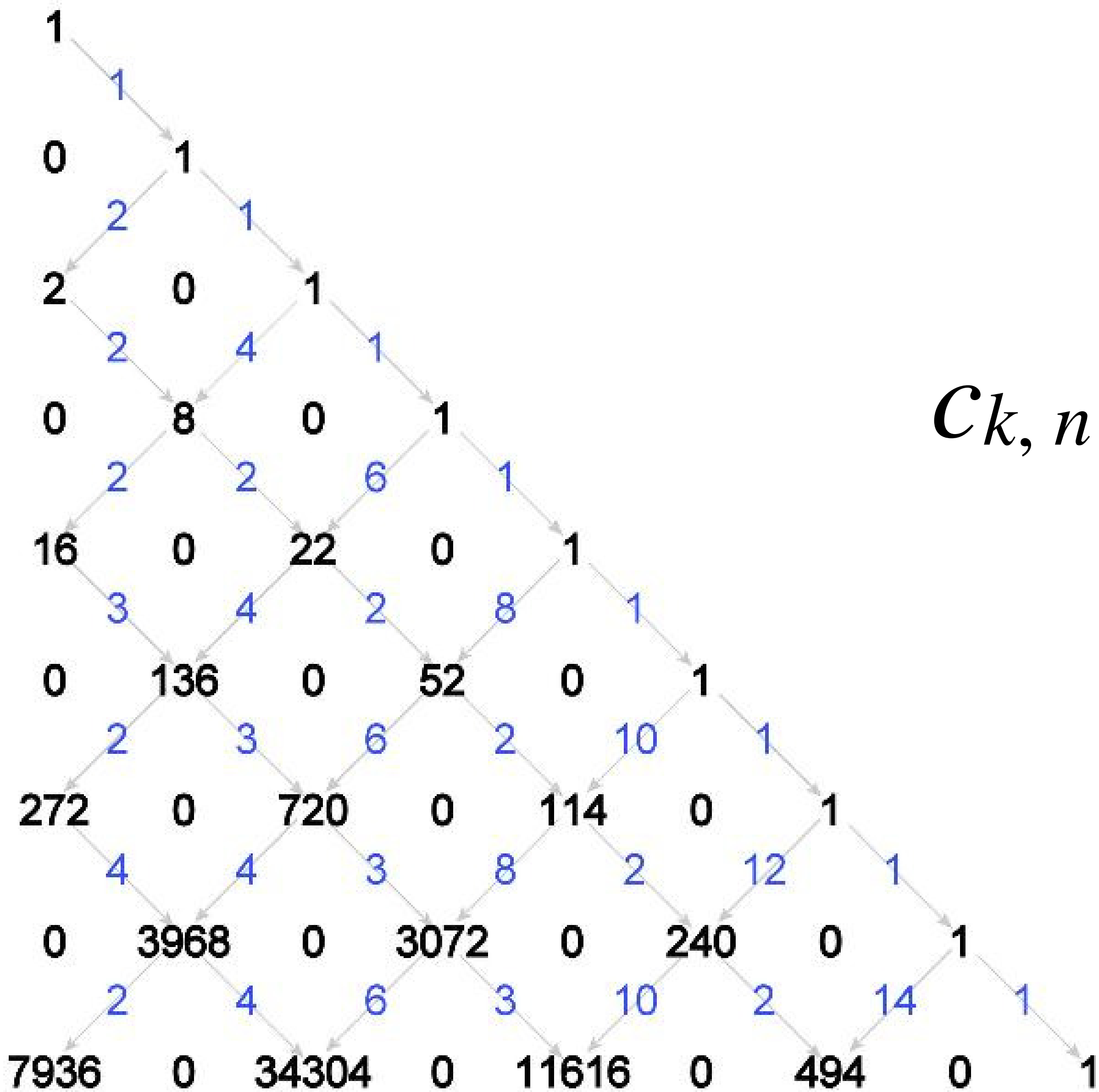
$$c_{k,n+1} = \frac{n-k+2}{2}c_{k-1,n} + 2(k+1)c_{k+1,n},$$

where we'll interpret $c_{-1,n}$ as 0.

Jasmine: It's kind of like a more complex version of Pascal's triangle. Let's make a diagram showing how the coefficients of $p_n(x)$ feed into the coefficients of $p_{n+1}(x)$.

Emily: Great idea!

Emily and Jasmine create a table of the coefficients of the polynomials $p_n(x)$. They write the coefficients of the polynomial in black. In blue over each arrow, they write the factor by which the coefficient at the base of the arrow contributes to the coefficient that the arrow points to. See the diagram at the top of the next page.



The coefficients of the polynomials $p_n(x)$ arranged in a triangular array.

The coefficients of $p_n(x)$ are written in the nth row of black numbers in order of lowest degree to highest. For example, the 34304 in the bottom row is the coefficient of x^2 in $p_9(x)$, and the recursion formula tells us that 34304 = 4(3968) + 6(3072).

Jasmine: I see. The arrows that point southeast are arranged in diagonals (that are infinitely long), and those arrows are weighted by the numbers 1, 2, 3, etc., counting from the uppermost diagonal and going down. On the other hand, the arrows that point southwest are arranged in diagonals (that are finite), and the arrows on each such diagonal are weighted by the consecutive even integers starting from the lowest arrow on the diagonal and going up and to the right.

Emily: The numbers we need to get the Taylor series for tangent are in the first column: 1, 0, 2, 0, 16, 0, 272, 0, 7936, …

Jasmine: Yes. That tells us that the Taylor series for tangent begins

$$x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \frac{272}{7!}x^7 + \frac{7936}{9!}x^9 + \ldots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$

Emily: It's curious that all the entries below the diagonal of 1's are even.

Jasmine: Yes, that's true because the 1's feed to the number down and to the left by a factor of 2, so all the entries in the diagonal just below the 1's will be even. And then the recursion formula shows that all the entries below that diagonal will also be even.

Emily: Wait a second… the entries in the second diagonal below the 1's all seem to be multiples of 4… and the entries in the third diagonal below the 1's are all multiples of 8…

Jasmine: Are you thinking that all the entries in the kth diagonal below the 1's are going to be multiples of 2^k?

Emily: Maybe!

Jasmine: So if we set $d_{k,n}$ to be $c_{k,n}/2^{(n-k+1)/2}$, you're saying that the $d_{k,n}$ will all be nonnegative integers?

Emily: Yes. Let's substitute $2^{(n-k+1)/2}d_{k,n}$ for $c_{k,n}$ in the recurrence relation for the coefficients. Hopefully, from that relation we'll be able to see that the $d_{k,n}$ are nonnegative integers.

Jasmine: Good idea! That substitution gives us

$$2^{(n+1-k+1)/2}d_{k,n+1} = \frac{n-k+2}{2}2^{(n-(k-1)+1)/2}d_{k-1,n} + 2(k+1)2^{(n-(k+1)+1)/2}d_{k+1,n},$$

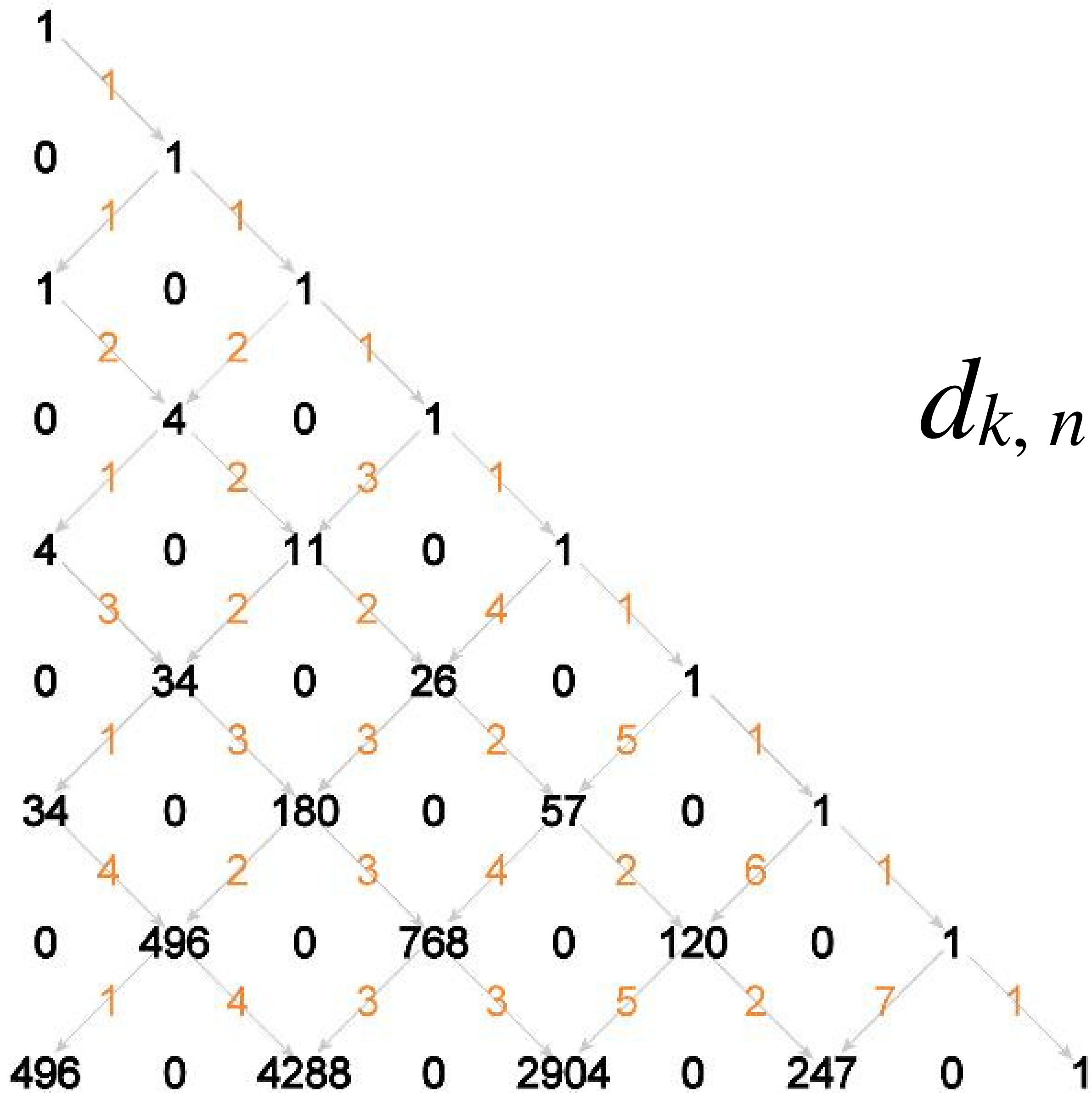
where, just as with the $c_{k,n}$, we assume that $d_{-1,n} = 0$ for all n. If we divide throughout by $2^{(n+1-k+1)/2}$, we get

$$d_{k,n+1} = \frac{n-k+2}{2}d_{k-1,n} + (k+1)d_{k+1,n}.$$

That's almost the same as the recurrence relation for the $c_{k,n}$! The only difference would be to halve all the entries on the flow arrows that point to the southwest in our "flow diagram" for the coefficients $c_{k,n}$. Since $d_{0,1} = 1$, that does mean that all the $d_{k,n}$ are nonnegative integers!

Emily and Jasmine make a "flow diagram" for the $d_{k,n}$. See the figure at the top of the next page.

Emily: Unfortunately, I don't see how we can compute the $d_{k,n}$ without using the recurrence relation. All this observation does, it seems, is to make the entries smaller, although some of the numbers have relatively large prime factors. This makes me think that unlike the entries in Pascal's triangle, there isn't going to be a simple expression for the $d_{k,n}$ as a product.



$d_{k,\,n}$

Jasmine: The sums of the entries in each row of Pascal's triangle give the powers of 2. I wonder if the sums of the entries in each row of our $d_{k,\,n}$ triangle yield some simple sequence.

Emily: Let's check! I get row sums of 1, 1, 2, 5, 16, 61, 272, 1385, 7936, …

Jasmine: Hold on! That's strange. For the odd rows, the sums are the nonzero entries in the first column of the triangle of $c_{k,\,n}$ values: 1, 2, 16, 272, 7936, …!

Emily: Whoa, that can't be a coincidence! What's going on here?

To be continued…

Anti-magic Squares[1]

by Robert Donley[2]
edited by Amanda Galtman

For this installment, we consider a new direction based on semi-magic squares. In the preceding series on graph theory, we saw that semi-magic squares with 0/1 entries become difficult as their size grows. For the case of size 3, the situation is tractable, and we introduce the anti-magic square as a variation with interesting counting properties.

To begin, we review semi-magic squares of size 3; much of this material appears in the preceding installments, but no graph theory will be needed for this new series. Our goal is to find a simpler model for such squares and to derive counting formulas based on line sums. It will be helpful to recall the binomial series formula and how generating functions work ("Fibonacci Numbers and Multiset Counting," Volume 15, Number 6, and "Generating Functions for Compositions," Volume 16, Number 4).

Definition: A **semi-magic square** of size n with **line sum** L is a square matrix of size n with nonnegative integer entries such that the sum along each row or column equals L.

Example: A square matrix with all entries equal to zero is semi-magic with $L = 0$.

We can also directly identify all semi-magic squares with line sum $L = 1$.

Definition: A **permutation matrix** is a square matrix having exactly one 1 in each row and column. All other entries vanish.

Exercise: Prove that a semi-magic square with line sum $L = 1$ must be a permutation matrix.

Example: The square matrix J_n of size n with all entries equal to 1 is a semi-magic square with line sum $L = n$.

Exercise: Prove that if M and N are semi-magic squares with line sums L and L', respectively, then $M + N$ is a semi-magic square with line sum $L + L'$.

Thus, the sum of any L permutation matrices is a semi-magic square with line sum L. If M is a semi-magic square with line sum L, then kM, the sum of k copies of M, is a semi-magic square with line sum kL.

Example: Consider the six permutation matrices of size 3:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

[1] This installment is 23rd in a series that began in Volume 15, Number 3. This installment is the first of a new subseries.

[2] This content is supported in part by a grant from MathWorks.

$$P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, P_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } P_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that, to obtain the latter three squares, we switch the last two columns of the first three squares. We see that

$$P_1 + P_2 + P_3 = P_4 + P_5 + P_6 = J_3,$$

and that J_3 has line sum $L = 3$.

The converse of the statement about permutation matrices is true.

Theorem (Birkhoff-von Neumann): Every semi-magic square with line sum L is a sum of L permutation matrices.

Before we prove this for size 3, we note that if M is a semi-magic square with line sum L, then permuting any rows or columns of M results in a semi-magic square with line sum L. A similar statement holds for the transpose of M.

Proof of theorem for size 3. The proof will require parts that we leave to the reader as exercises. Fix a positive L. Suppose the statement is true for all semi-magic squares with line sum less than L. If every entry of M is positive, then we can subtract the matrix J_3 from M to obtain a semi-magic square M' with line sum $L - 3$, which in turn can be written as a sum of $L - 3$ permutation matrices. Thus

$$M = M' + P_1 + P_2 + P_3$$

is a sum of L permutation matrices. Now show the following:

Exercise: Let M be a semi-magic square of size 3 with positive line sum L. Prove that there exists some permutation of the rows and columns of M such that the new diagonal entries are all positive.

Definition: We call two permutation matrices P and Q **deranged** if their sum has entries with value at most 1. That is, such matrices share no entries of 1 in the same position.

Exercise: In the previous exercise, suppose M has at least one vanishing entry. Prove that we can obtain a new semi-magic square M'' with no vanishing entries by adding two deranged permutation matrices to M.

To finish the proof, if M has some vanishing entries, then, for some deranged permutation matrices P and Q, $M + P + Q$ has no vanishing entries and line sum $L + 2$. Now M' defined as $M + P + Q - J_3$ has line sum $L - 1$ and is a sum of $L - 1$ permutation matrices. But $P' = J_3 - P - Q$ is a permutation matrix, so $M = M' + P'$ is a sum of L permutation matrices. □

By the Birkhoff-Von Neumann theorem, any semi-magic square M of size 3 may be expressed in the form

$$M = a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5P_5 + a_6P_6,$$

where each $a_i \geq 0$. An important advantage of this representation is that, as long as we remember the ordering of the permutation matrices, we can simply describe M as an ordered sextuple $\boldsymbol{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$. Note that the line sum of M is the sum of the entries of $\boldsymbol{a}$.

It is advantageous to configure our notation by arranging the sextuple in a rectangle with two rows, as seen on the right.

a_1	a_2	a_3
a_4	a_5	a_6

One disadvantage is that this representation need not be unique; that is, there may be many ways to describe M as a rectangle. For instance, J_3 is described by both rectangles at right:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

Exercise: Suppose the sextuple $\boldsymbol{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$ corresponds to M. Prove that

$$\boldsymbol{a'} = (a_1 + 1, a_2 + 1, a_3 + 1, a_4 - 1, a_5 - 1, a_6 - 1)$$

also corresponds to M when all entries of $\boldsymbol{a'}$ are nonnegative.

This shifting of 1s is the only defect in uniqueness, and we visualize this defect in the rectangle notation as shifting triples of 1s between rows. Suppose M is represented by both

$$\boldsymbol{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \text{ and } \boldsymbol{a'} = (a'_1, a'_2, a'_3, a'_4, a'_5, a'_6).$$

Then,

$$M = a_1P_1 + \ldots + a_6P_6 = a'_1P_1 + \ldots + a'_6P_6,$$

or

$$(a_1 - a'_1)P_1 + \ldots + (a_6 - a'_6)P_6 = 0.$$

Exercise: Suppose $b_1P_1 + \ldots + b_6P_6 = 0$. Prove that $b_1 = b_2 = b_3 = -b_4 = -b_5 = -b_6$ by rewriting the equation as matrices of size 3.

Exercise: From the previous exercise, prove that the only way two sextuples can represent the same M is if they differ by one or more shiftings of 1s as above.

To summarize the preceding discussion, we highlight three key points:

- $\boldsymbol{a}$ and $\boldsymbol{a'} = (a_1 + k, a_2 + k, a_3 + k, a_4 - k, a_5 - k, a_6 - k)$ represent the same M for all k when $\boldsymbol{a'}$ has all entries nonnegative,
- two sextuples corresponding to M must be of this form, and
- we obtain a preferred representative by upshifting triples of 1s until one of a_4, a_5, or a_6 vanishes.

It follows immediately that M has a unique representation when one of a_1, a_2, or a_3 vanishes in the preferred representation. Such squares M have at least one vanishing entry, and the corresponding rectangles have at least one vanishing entry in each row.

Exercise: Express the following semi-magic squares as sextuples and rectangles. List all possible rectangles when not unique.

$$M_1 = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 5 \\ 4 & 3 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, M_3 = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}.$$

Exercise: Find a column operation on M that interchanges the rows of a rectangle. How does transpose on M affect a rectangle?

Exercise: List all 21 semi-magic squares of size 3 and line sum $L = 2$. Partition these squares into subsets by the number of entries with value 2, and find the corresponding sextuples. For each fixed subset, how are the sextuples of the squares in that subset related?

In the previous exercise, there are three representative rectangles:

1	1	0
0	0	0

1	0	0
1	0	0

2	0	0
0	0	0

Exercise: Reconcile the count of 21 using the formula: $21 = (3 + 3) + (3 \times 3) + 6$.

Exercise: Repeat the previous analysis for M with line sum $L = 3$. In this case, the only square with a non-unique representative is J_3.

Exercise: Repeat the previous analysis for M with line sum $L = 4$.

As usual, we denote the binomial coefficient "n choose k" by $\binom{n}{k}$, and, as a convention, we suppose this coefficient vanishes if k is negative or greater than n.

Theorem: Let $\rho(L)$ be the number of semi-magic squares of size 3 with line sum L. Then

$$\rho(L) = \binom{L+5}{5} - \binom{L+2}{5} = \frac{(L+2)(L+1)(L^2+3L+4)}{8}.$$

Proof. Recall that the number of ways to put L balls into n boxes is given by the binomial coefficient $\binom{L+n-1}{n-1}$. If we consider the preferred representation of M as a rectangle with vanishing of one of the lower entries, then $\rho(L)$ is the number of ways to put L balls into six boxes, except that we discard any sextuple in which all lower entries are positive. To construct a rectangle of the latter type, we pre-assign one ball to each of these entries and then place $L - 3$ balls into the six boxes in any manner. The first formula now follows. For the second formula, we expand the binomial coefficients into factorials and simplify. □

Exercise: Prove the second formula for $\rho(L)$.

Exercise: Use the second formula to calculate $\rho(L)$ for $0 \le L \le 6$. Search for this sequence in the Online Encyclopedia of Integer Sequences (oeis.org).

Next, we find the generating function associated to $\rho(L)$, which is the formal power series

$$F(x) = \rho(0) + \rho(1)x + \rho(2)x^2 + \rho(3)x^3 + \ldots.$$

Corollary: Let $\rho(L)$ be the number of semi-magic squares of size 3 with line sum L. Then the generating function for $\rho(L)$ is given by

$$F(x) = \frac{1-x^3}{(1-x)^6} = \frac{1+x+x^2}{(1-x)^5}.$$

Proof. The corollary follows from the formula for the binomial series. □

From the second formula in the corollary, we obtain another formula for $\rho(L)$, which again follows from the formula for the binomial series.

Corollary: Let $\rho(L)$ be the number of semi-magic squares of size 3 with line sum L. Then

$$\rho(L) = \binom{L+4}{4} + \binom{L+4}{3} + \binom{L+4}{2}.$$

Exercise: Prove the third formula using the binomial series.

Exercise: Use the third formula to calculate $\rho(L)$ for $0 \le L \le 6$. Then adapt the proof of the theorem to give a counting argument for this formula. Instead of discarding cases without zeros, model your argument on how to place zeros in the last three entries.

Let's summarize what we have so far:

- a preferred representation for semi-magic squares of size 3 as sextuples or rectangles, and
- several formulas for the number of semi-magic squares of size 3 with line sum L.

We might hope that our arguments extend to semi-magic squares of size 4, but the difficulties with semi-magic squares of size 4 and larger have been noted and explored in the series preceding this installment.

Exercise: Write the matrix $D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ as a sum of permutation matrices in four distinct ways.

To find a set of matrices that extends the sextuple or rectangle models, we instead consider anti-magic squares, which we now define.

Definition: Fix a positive integer n. Let R_i be the square matrix of size n for which the entries in the ith row all equal 1 and all other entries vanish. Likewise, define C_i in a similar manner, but for the ith column.

Definition: The set of anti-magic squares of size n consists of all sums of matrices of type R_i and C_i. That is, an **anti-magic square** M is a square matrix of the form

$$M = r_1R_1 + \ldots + r_nR_n + c_1C_1 + \ldots + c_nC_n,$$

with integers r_i, $c_i \geq 0$. We define the **index** of M to be the sum

$$i(M) = r_1 + \ldots + r_n + c_1 + \ldots + c_n.$$

Example: The anti-magic squares of size 3 are sums of the matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Exercise: Prove that the index of M is the sum of the diagonal entries of M.

Exercise: How many anti-magic squares have index 1? Index 2 through n?

We see that the set of anti-magic squares is preserved under permutations of rows and columns, and transpose interchanges R_i and C_i. Of course, the index is unchanged under these operations. Furthermore, most of the counting results for semi-magic squares of size 3 carry over as a consequence of the relation

$$R_1 + \ldots + R_n = C_1 + \ldots + C_n = J_n.$$

Exercise: Repeat the sequence of exercises for semi-magic squares of size 3 to prove that every anti-magic square of size n is uniquely represented by a rectangle with width n and nonnegative entries such that at least one c_i vanishes.

r_1	. . .	r_3
c_1	. . .	c_3

The following exercise appears as problem 53(c) in Chapter 4 of Richard Stanley's book Enumerative Combinatorics, Volume 1.[3]

Exercise: Prove that the number of anti-magic squares of size n and index L is given by the formula

[3] Stanley, R.P., 2011. Enumerative combinatorics, Volume 1, 2nd edition. *Cambridge studies in advanced mathematics.*

$$\rho'(L) = \binom{L+2n-1}{2n-1} - \binom{L+n-1}{2n-1}.$$

Exercise: Find a generating function for $\rho'(L)$ and a second formula for $\rho'(L)$ based on the third formula for $\rho(L)$. Find the first 10 terms when $n = 2$ and $n = 4$.

We finish with a tabular method to calculate $\rho'(L)$. Recall the geometric series formula

$$P(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.$$

If $G(x) = s_0 + s_1x + s_2x^2 + \ldots$ is the generating function for the sequence s_m, then $P(x)G(x)$ is the generating function for the sequence t_k of partial sums of s_m, where

$$t_k = s_0 + s_1 + \ldots + s_k.$$

When $n = 3$, the numerator of $\rho(L)$ is $G(X) = 1 + x + x^2$, for which we record the coefficients with increasing degree r in column 0 of the table below. Each subsequent column s records the entries of $P(x)^s G(x)$. To implement the partial sum operation, we add the entries in a column down to a given position and place the sum in the cell to the right.

Alternatively, we add an entry in a given position to the entry one down and to the left, and we place the sum in the position below the original position. For instance, in column 3, 19 + 12 =31. Column 5 gives the list of values for $\rho(L)$.

$r \backslash s$	**0**	**1**	**2**	**3**	**4**	**5**
0	1	1	1	1	1	$\boxed{1}$
1	1	2	3	4	5	$\boxed{6}$
2	1	3	6	10	15	$\boxed{21}$
3	0	3	9	$\boxed{19}$	34	$\boxed{55}$
4	0	3	$\boxed{12}$	$\boxed{31}$	65	$\boxed{120}$
5	0	3	15	46	111	$\boxed{231}$
6	0	3	18	64	175	$\boxed{406}$

Exercise: Explain why the three-term partial sum calculation works. Then use the tabular method to calculate the first 10 values of $\rho'(L)$ for $n = 2, 4, 5$.

Hypercube Sections: Combinatorics

by Addie Summer | edited by Amanda Galtman

Let n be a positive integer greater than 1, and let k be an integer such that $0 < k < n$.

Last time,[1] we found that the hypercube cross section $C_{k,\,n}$ has $2n$ $(n-2)$-dimensional faces, unless $k = 1$ or $k = n - 1$. In those cases, $C_{k,\,n}$ is an $(n-1)$-dimensional simplex, which has n $(n-2)$-dimensional faces.

We also found that $C_{k,\,n}$ has:

$$ {}_nC_k = \frac{n!}{k!(n-k)!} \text{ vertices,} $$

$$ \frac{n!}{2(k-1)!(n-k-1)!} \text{ edges,} $$

$$ \text{and } \frac{n!(n-2)}{6(k-1)!(n-k-1)!} \text{ triangular faces.} $$

Technically, we derived the formula for the number of faces under the assumption that $1 < k < n - 1$. But note that when $k = 1$ or $k = n - 1$, the object $C_{k,\,n}$ is an $(n-1)$-dimensional simplex, which has ${}_nC_3$ triangular faces (since any three of its n vertices form a triangular face). The last formula does, indeed, equal ${}_nC_3$ when $k = 1$ or $k = n - 1$.

Let's press on!

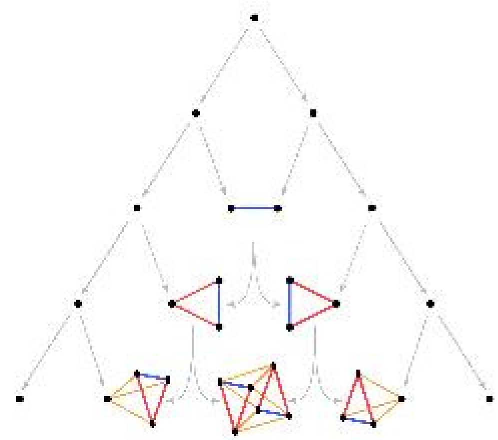
Instead of counting 3D faces of $C_{k,\,n}$, I feel like tackling the general case. We've seen that the geometric shapes appearing in our geometric version of Pascal's triangle in the rows above the nth are the possible faces of $C_{k,\,n}$. So let's ask: how many faces of $C_{k,\,n}$ are congruent to $C_{j,\,p}$, where $p < n$ and $0 < j < p$? (The latter inequality ensures that $C_{j,\,p}$ is a $(p-1)$-dimensional object and not a point.)

We have to be mindful that $C_{j,\,p}$ is congruent to $C_{p-j,\,p}$. Please keep this symmetry of Pascal's triangle in mind in the following discussion.

As we saw, we can find all the $(p-1)$-dimensional faces of $C_{k,\,n}$ by looking at its intersection with various p-dimensional faces of the hypercube. We can obtain all these p-dimensional faces of the hypercube by first picking a set I of p indices (between 1 and n, inclusive). Then, for i not

Here, Addie takes the n-dimensional hypercube to be the points in n-dimensional space with coordinates $(x_1, x_2, x_3, \ldots, x_n)$, where $-1 \le x_k \le 1$ for $k = 1, 2, 3, \ldots, n$. Its vertices are the points each of whose coordinates is either 1 or -1.

For each integer k from 0 to n, let $V_{k,\,n}$ be the set of vertices that have exactly k coordinates equal to -1, and let $C_{k,\,n}$ be the convex hull of the vertices in $V_{k,\,n}$. If it is clear what n is, Addie might omit the n and simply write V_k and C_k.



This is the geometric version of Pascal's triangle we found last time. Each shape is the "prism" formed by using the two shapes directly above as bases. The nth row gives snapshots of cross sections of the n-dimensional hypercube by an $(n-1)$-dimensional hyperplane passing through. The number of vertices of each shape is given by Pascal's triangle.

[1] "Hypercube Sections: Pascal Revisited," Volume 18, Number 4.

in I, we set each x_i independently to 1 or -1. For i in I, we let x_i range freely over the interval [-1, 1]. We called the indices in I the "free" indices. Let's call the indices outside I the "fixed" indices. We also found it convenient to define S to be the sum of x_i over the fixed indices i and saw that S has the same parity as $n - p$.

The points of this p-dimensional hypercube that intersect the cross sectioning hyperplane

$$x_1 + x_2 + x_3 + . . . + x_n = n - 2k$$

(that "slices out" $C_{k,\,n}$) are those for which $\sum_{i \in I} x_i = n - 2k - S$. Usually, there are *two* ways for these points to correspond to a face congruent to $C_{j,\,p}$, because of the "Pascal symmetry" mentioned earlier (i.e., that $C_{j,\,p}$ is congruent to $C_{p-j,\,p}$). If $j \neq p - j$, i.e., $p \neq 2j$, then one way is for $n - 2k - S = p - 2j$, and the other way is for $n - 2k - S = 2j - p$. If $p = 2j$, then there is only one way; these points must correspond to a face congruent to $C_{j,\,2j}$.

To avoid confusion, let's count only the ways that correspond to $n - 2k - S = p - 2j$ or, after rearranging, $S = n - p - 2(k - j)$. Later, we'll account for the "other way" by combining the result with the formula obtained by substituting $p - j$ for j, in case $p \neq 2j$. Let's also say that a face is **strictly congruent** to $C_{j,\,p}$ if and only if that face is the cross section of a p-dimensional hypercube facet exactly in this way, with $S = n - p - 2(k - j)$.

Recall that S is the sum of the fixed coordinates x_i. These coordinates are equal to +1 or -1, and since there are $n - p$ of these fixed coordinates, it must be that $|S| \leq n - p$. Therefore, the condition $S = n - p - 2(k - j)$ implies

$$-(n - p) \leq n - p + 2(j - k) \leq n - p.$$

The first inequality is equivalent to $p - j \leq n - k$, and the last inequality is equivalent to $j \leq k$. This makes sense because if a vertex of $C_{k,\,n}$ is a vertex of a $(p - 1)$-dimensional face contained in the p-dimensional hypercube with free indices I, then we need exactly j of these free indices to correspond to coordinates set to -1. The remaining $p - j$ free indices correspond to coordinates set to +1. At the same time, a total of k of the vertex's coordinates must be -1, while the remaining $n - k$ of them must be +1. This is possible only if $k \geq j$ and $n - k \geq p - j$.

Let's assume the inequalities $k \geq j$ and $n - k \geq p - j$ hold (and also that $0 < k < n$ and $0 < j < p$). Fix a vertex v in $C_{k,\,n}$. We know that k of its coordinates are equal to -1 and $n - k$ of its coordinates are equal to +1.

How many faces have v as a vertex and are strictly congruent to $C_{j,\,p}$ with $n - 2k - S = p - 2j$?

There are as many as the number of ways to pick j of the k coordinates of v that are equal to -1 multiplied by the number of ways to pick $p - j$ of the $n - k$ coordinates of v that are equal to +1:

$${}_kC_j \cdot {}_{n-k}C_{p-j}.$$

If we interpret ${}_aC_b$ as 0 if $b < 0$ or $b > a$, then this formula is valid for all $0 < k < n$, so let's agree to do that!

There are ${}_nC_k$ vertices of $C_{k,\,n}$, but if we add up ${}_kC_j \cdot {}_{n-k}C_{p-j}$ over all vertices, we will count each face strictly congruent to $C_{j,\,p}$ (with $n - 2k - S = p - 2j$) a total of ${}_pC_j$ times. Therefore, the number of faces strictly congruent to $C_{j,\,p}$ with $n - 2k - S = p - 2j$ is equal to

$$\frac{{}_nC_k \cdot {}_kC_j \cdot {}_{n-k}C_{p-j}}{{}_pC_j}.$$

Let's see if we can simplify this expression.

$$\begin{aligned}\frac{{}_nC_k \cdot {}_kC_j \cdot {}_{n-k}C_{p-j}}{{}_pC_j} &= \frac{n!}{k!(n-k)!}\frac{k!}{j!(k-j)!}\frac{(n-k)!}{(p-j)!(n-k-p+j)!}\frac{j!(p-j)!}{p!} \\ &= \frac{n!}{p!(k-j)!(n-k-p+j)!}\end{aligned}$$

Notice that $p + (k - j) + (n - k - p + j) = n$. That means that this expression corresponds to a trinomial coefficient. Specifically, it is the coefficient of $x^py^{k-j}z^{n-k-p+j}$ in the expansion of $(x + y + z)^n$. Multiplying the numerator and denominator by $(n - p)!$, we can also express this as:

$$\frac{n!(n-p)!}{p!(n-p)!(k-j)!(n-k-p+j)!} = {}_nC_p \cdot {}_{n-p}C_{k-j}.$$

I wonder if this expression can be interpreted in terms of the geometry. We're finding the faces that are strictly congruent to $C_{j,\,p}$ by intersecting $C_{k,\,p}$ with p-dimensional hypercubes. So perhaps the first factor, ${}_nC_p$, can be interpreted as choosing our p free coordinates I. Having chosen the free coordinates, we then need to set each fixed coordinate to +1 or -1 in such a way that they sum to $S = n - p - 2(k - j)$. This means that of the $n - p$ fixed coordinates, exactly $k - j$ of them must be set to -1…so that accounts for the second factor!

The product ${}_nC_p \cdot {}_{n-p}C_{k-j}$ is nonzero only if $n \geq p$ and $n - p \geq k - j$. These are the same inequalities that we found earlier for the corresponding p-dimensional hypercube face to contain a $(p - 1)$-dimensional face of $C_{k,\,n}$ that is strictly congruent to $C_{j,\,p}$!

Notice that ${}_nC_p \cdot {}_{n-p}C_{k-j} = 1$ when $n = p$ and $j = k$. This makes sense because it's the number of faces of $C_{k,\,n}$ that are congruent to $C_{k,\,n}$. There's only one, namely, itself!

To summarize, the number of faces of $C_{k,\,n}$, where $0 < k < n$, that are *strictly* congruent to $C_{j,\,p}$, where $0 < j < p$, is given by ${}_nC_p \cdot {}_{n-p}C_{k-j}$, provided that $n \geq p$ and $n - p \geq k - j$. This formula covers all faces except the vertices. But we know that there are ${}_nC_k$ vertices of $C_{k,\,n}$.

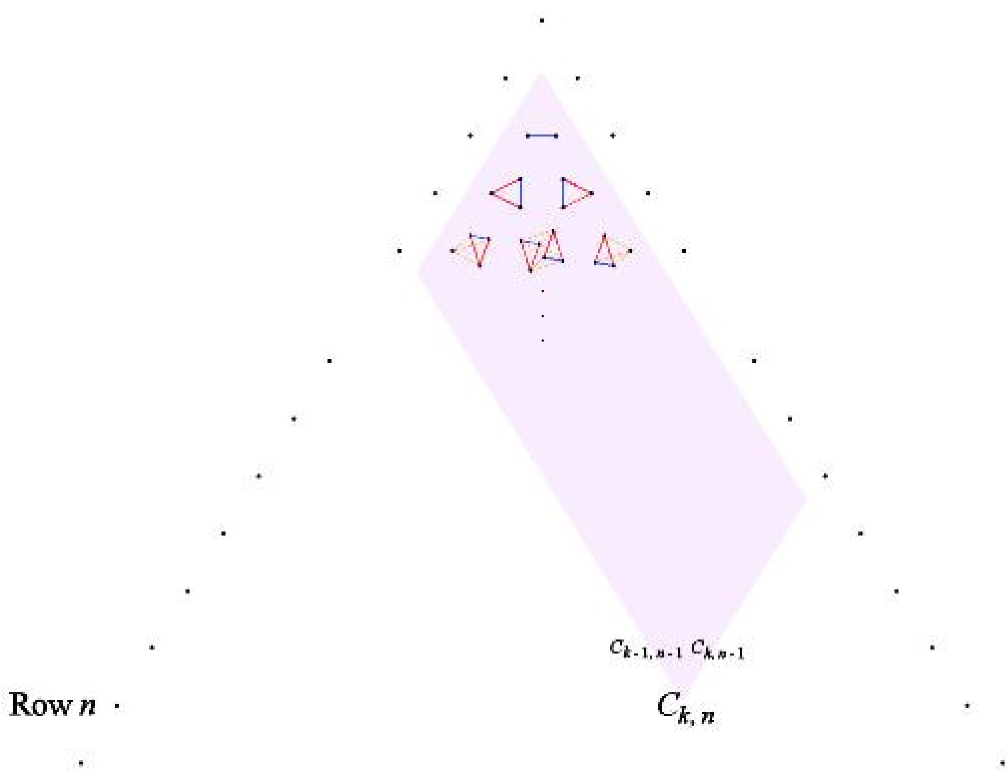
If $p \neq 2j$, then the number of faces of $C_{k,\,n}$ that are congruent to $C_{j,\,p}$ is

$${}_nC_p({}_{n-p}C_{k-j} + {}_{n-p}C_{k-p+j}) = {}_nC_p({}_{n-p}C_{k-j} + {}_{n-p}C_{n-k-j}).$$

Let's see if we can develop a clearer picture of which facets occur in $C_{k,\,n}$. If $C_{j,\,p}$ is a facet of $C_{k,\,n}$, where $0 < j < p$, we have seen that $n \geq p$ and $n - p \geq k - j$. The first inequality says that the facets of $C_{k,\,n}$ appear above or in the same row as $C_{k,\,n}$, which makes sense because the objects in the rows below are higher-dimensional objects. In order for ${}_nC_p \cdot {}_{n-p}C_{k-j}$ to be positive, we must have $k - j \geq 0$, so $n - p \geq k - j \geq 0$. If we think of p as fixed, the values of j that satisfy this compound inequality correspond to the $C_{j,p}$ that are facets of $C_{k,n}$. So let's rearrange these inequalities to isolate j. The first inequality, $n - p \geq k - j$, rearranges to $j \geq k + p - n$. The second inequality, $k - j \geq 0$, rearranges to $j \leq k$. Thus,

$$k + p - n \leq j \leq k.$$

This compound inequality bounds j between lines in Pascal's triangle that are parallel to its edges and pass through the kth entry in row n. Combining this with the requirement that $0 < j < p$, we can illustrate with this figure:



The shaded parallelogram covers those $C_{j,p}$, $0 < j < p$, that appear as facets of $C_{k,n}$ and for which the formula for the number of such facets that are strictly congruent to $C_{j,p}$ applies. The only missing facets are the vertices, of which there are ${}_nC_k$. For vertices, our formula works only when $p = j = 0$.

Combinatorics of $C_{2,5}$

Among the 4D cross sections, up to congruence, there are only two: $C_{1,5}$ and $C_{2,5}$. We know that $C_{1,5}$ is a 4D regular simplex with five vertices, 10 edges, 10 triangular faces, and five tetrahedral 3D faces. For fun, let's apply what we've learned to get to know $C_{2,5}$ better.

According to our analysis, faces of $C_{2,5}$ are regular tetrahedra ($C_{1,4}$), regular octahedra ($C_{2,4}$), equilateral triangles, line segments (all the same length, as is true for all $C_{k,n}$), and vertices.

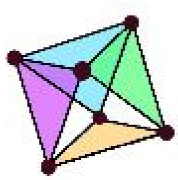
Using our formula, there are

$${}_5C_4 \cdot {}_1C_1 = 5 \text{ regular tetrahedral faces}$$
$${}_5C_4 \cdot {}_1C_0 = 5 \text{ regular octahedral faces}$$
$${}_5C_3 \cdot {}_2C_1 + {}_5C_3 \cdot {}_2C_0 = 30 \text{ equilateral triangular faces}$$
$${}_5C_2 \cdot {}_3C_1 = 30 \text{ edges}$$
and ${}_5C_2 = 10$ vertices.

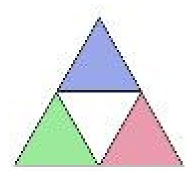
In 5D Euclidean space, the vertices of $C_{2,5}$ are the 10 points whose coordinates are permutations of (1, 1, 1, -1, -1). Each vertex has six edges emanating from it since each edge has two vertices. But if we multiply the number of edges by 2, we will overcount the vertices exactly by the number of edges emanating from each vertex. Similarly, each vertex is a vertex of nine triangular faces, and each edge is an edge of three equilateral triangular faces. Let v be a vertex, and let t be the number of tetrahedral faces that v is a vertex of. Since a tetrahedron has four vertices, adding up all the vertices for each tetrahedron will overcount the number of tetrahedra per vertex by a factor of t. That is, $4(5)/t = 10$. Hence, every vertex is a vertex of two tetrahedral faces. A similar argument shows that every vertex is the vertex of three octahedral faces.

How do these five 3D faces at a given vertex meet each other? Suppose we have two octahedral faces. From our earlier analysis, we know that each is the intersection of a 4D-hypercubic face with $C_{2,5}$. Picking a 4D-hypercubic face means to fix one of the five coordinates and let the other four be free. Furthermore, we must set the fixed coordinate to +1, or else we have a tetrahedral face. So the two octahedral faces are specified by a choice of two coordinates, which must both equal +1. The vertices common to both octahedrons have +1 on these two coordinates. The remaining three coordinates have two -1s and one +1, forming the three vertices of a shared triangular face. Similar reasoning applies to the other pairings to show that every pair of these five 3D faces that meet at a common vertex share a triangular face, except for the two tetrahedra. In fact, *any* two tetrahedral faces intersect in only a vertex, but tetrahedra and octahedra always share a single triangular face, as do any pair of octahedra (see the cover).[2]

Imagine: we have five regular tetrahedra in 4D space, every pair touching at a vertex, the 10 points of contact being all the vertices of $C_{2,5}$. If we restrict our imagination to 3D, such a configuration of five tetrahedra appears impossible. How can we imagine such a structure?



There's an analog of this structure of five tetrahedra in $C_{2,4}$, the regular octahedron, whose faces consist of four triangles *strictly* congruent to $C_{1,3}$ and four strictly congruent to $C_{2,3}$. The four triangles strictly congruent to $C_{1,3}$ form the analogous structure. Here, four equilateral triangles touch pairwise at a vertex, and the six points of contact among them form the vertices of the octahedron. See the figure at left, where the four equilateral triangles are colored.



If a creature confined to the plane contemplated such a configuration of triangles, that creature would probably think it impossible. The creature would be able to place three triangles with each pair sharing a vertex, as shown at right, but would have a hard time imagining placing a fourth triangle, of the same size, so that it touches these three at their "free" vertices. What the flat creature is missing is the

[2] $C_{2,5}$ is also known as a **rectified 5-cell**.

ability to imagine in 3D, where the three triangles can hinge around the edges of the central hole, rotating until their three free vertices form the vertices of that fourth equilateral triangle.

What we, as 3D creatures, might imagine, then, is to start by placing four identical regular tetrahedra onto the four faces of a regular tetrahedral "hole," something we can actually do in real life. Each tetrahedron touches the other three and has a fourth "free" vertex. In this pyramidal configuration, those four free vertices form a regular tetrahedron, but one that is bigger than the others. Then, imagine that these four tetrahedra hinge around the triangular faces of the hole into the fourth dimension, bringing their free vertices closer until those free vertices form a tetrahedron congruent to the original four. In 4D, you can rotate around planes!

Combinatorics of $C_{3,6}$

Let's briefly visit the 5D object $C_{3,6}$. Without my explaining, can you verify the following facts?

According to our analysis, faces of $C_{3,6}$ are itself, the $C_{2,5}$ and $C_{3,5}$ shapes we just studied (which are congruent), the regular tetrahedra $C_{1,4}$ and $C_{3,4}$, the regular octahedron $C_{2,4}$, the equilateral triangles $C_{1,3}$ and $C_{2,3}$, congruent edges $C_{1,2}$, and vertices.

Using our counting formula, we find that there are:

$_6C_5 \cdot {}_1C_1 + {}_6C_5 \cdot {}_1C_0 = 12$ faces congruent to $C_{2,5}$
$_6C_4 \cdot {}_2C_2 + {}_6C_4 \cdot {}_2C_0 = 30$ regular tetrahedral faces
$_6C_4 \cdot {}_2C_1 = 30$ regular octahedral faces
$_6C_3 \cdot {}_3C_2 + {}_6C_3 \cdot {}_3C_1 = 120$ equilateral triangular faces
$_6C_2 \cdot {}_4C_2 = 90$ edges
and $_6C_3 = 20$ vertices.

Every 4D face is attached to five other 4D faces along a tetrahedron and five along an octahedron. Every 4D face is on the opposite side of $C_{3,6}$ from another 4D face, with which it shares no vertices. More precisely, two 4D faces that are both strictly congruent to $C_{2,5}$ or to $C_{3,5}$ intersect in a tetrahedron, whereas the intersection of a 4D face strictly congruent to $C_{2,5}$ with one strictly congruent to $C_{3,5}$ is either an octahedron or empty. (And so each 4D face is joined by a tetrahedron to the other five faces it is strictly congruent to, is joined by an octahedron to five of the six faces it is not strictly congruent to, and is opposite the sixth of those faces.)

Every vertex has nine edges emanating from it and is a vertex of 18 triangles, nine octahedral faces, six tetrahedral faces, and six faces congruent to $C_{2,5}$, of which half are strictly congruent to $C_{2,5}$ and half to $C_{3,5}$.

Every edge is an edge of four triangular faces, four octahedral faces, two tetrahedral faces, and four faces congruent to $C_{2,5}$.

Every triangular face is a face of one tetrahedral face, two octahedral faces, and three faces congruent to $C_{2,5}$.

5D creatures must regard $C_{3,6}$ as one of the most beautiful objects to behold![3]

[3] $C_{3,6}$ is also known as a **birectified 5-simplex** or **dodecateron**.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 37 - Meet 1 September 11, 2025	Mentors:	Elisabeth Bullock, Clarise Han, Layla Jarrahy, Minerva Johar, Shauna Kwag, Yaqi Li, Maya Robinson, Ella Wilson, Dora Woodruff

We welcome all new and returning members and mentors to the start of our 19th year of Girls' Angle!

Tournament design offers a wide array of math problems. Suppose you have to design a fencing tournament with 10 competitors. You want every competitor to play every other competitor at least once and you organize the event into rounds. In each round, up to 4 matches are played simultaneously. What's the most efficient way to organize the matchups? What is the minimum number of rounds needed to meet all the requirements of the event? Can you generalize your results to N competitors where each round has up to M matches occurring simultaneously?

Now suppose you add a twist: Suppose every competitor needs some rest time, so no competitor may be scheduled to play four matches in a row. How does this requirement affect your schedules?

Session 37 - Meet 2 September 18, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Layla Jarrahy, Minerva Johar, Shauna Kwag, Hanna Mularczyk

Sometimes, if you're in the mood, plain old computation can be a lot of fun. Some members played around with finding fast ways to convert fractions to decimals. After converting several fractions to decimals, they noticed patterns that they could use to make their conversions even faster. For example, convert 17/99, 38/99, and 80/99 to decimals. Do you see a pattern? How much can you generalize this pattern? For example, after seeing the decimal expansions of 17/99, 38/99, and 80/99, what do you think the decimal expansion of 500/999 is? Can you prove the patterns you see work? Can you use what you notice to quickly convert 50/111 to a decimal? How about 2/37? And what about 25/101? For each of these fractions, you might notice something that you can exploit to be able to write down the decimal expansions with hardly any computational work.

Session 37 - Meet 3 September 25, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Clarise Han, Layla Jarrahy, Shauna Kwag, Yaqi Li, Hanna Mularczyk, Maya Robinson, Dora Woodruff

Take the month, day, and year of your birth and put them into the top three squares of a 3 by 3 grid. Can you fill in the 6 empty squares in this grid in such a way that the resulting array of 9 numbers is a magic square? That is, all the rows, columns, and major diagonals have entries that add up to the same constant?

For example, here's an array where the top row represents the date of release of the electronic version of this Bulletin:

10	31	25
37	22	7
19	13	34

No matter what the three numbers are in the top row, is it always possible to complete the array to form a magic square? Will it ever happen that there are multiple ways to complete the array to form a magic square for a given triple of numbers in the top row? What happens if you place the given numbers across the middle row? Will all the numbers necessarily be whole numbers?

Session 37 - Meet 4 October 2, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Clarise Han, Layla Jarrahy, Yaqi Li, Hanna Mularczyk, Dora Woodruff

Take your favorite geometric shape. How would you define it? A proper definition would be a precise description of the shape such that every object of that shape fits the description, but no object that does not have that shape fits the description. How would you define circle, square, isosceles triangle, or cube?

Mathematics demands precision. Every object of study in mathematics has been given a precise definition (or, is in the process of being given one). Having precise definitions enables us to prove things about mathematical objects and enables mathematicians to find agreement on the properties of objects. If two mathematicians disagree on a property of a mathematical object, all they have to do to figure out who is correct is to prove which property is consistent with the definition of the object and which is not. (While this may be easier said than done, it is always the case that either an object satisfies a property, or it does not, or the property is not relevant to the object, or one can choose whether or not the object has the property.)

Session 37 - Meet 5 October 9, 2025	Mentors:	Elisabeth Bullock, Clarise Han, Layla Jarrahy, Yaqi Li, Dora Woodruff
	Visitor:	Ila Fiete, MIT McGovern Institute

Ila Fiete is a Professor of Brain and Cognitive Sciences at MIT's McGovern Institute. She earned her bachelor's degree from the University of Michigan and her PhD in physics from Harvard.

Some years ago, scientists reported on the existence of cells in the brain that periodically fired when mice walked about. Prof. Fiete felt that those cells must be a part of network of cells and she wanted to find that network and create a model for it. Her idea initiated a multi-year research program that led to much deeper understanding of how many animals create spatial representations and navigate the world. Her work involves machine learning, neuroscience, and coding theory.

She began by showing us some animal's feats of navigation, such as the kangaroo rat navigating its complex network of tunnels or a type of ant in Morocco, *Cataglyphis fortis*, foraging for food. As the *C. fortis* searches for food, it traces out a seemingly random walk. But once it finds food, it heads straight back to its home. Interestingly, if the ant is put in a box and lifted from the food to some other place, when released, the ant will travel along a straight-line path parallel to the path from food source to home. When it finds no home where home should be, it commences on a systematic outwardly spiraling search pattern to find home.

When it comes to humans, navigation has been a great source of challenge that induced much innovation. Our modern Global Positioning System (GPS) required construction of rocket technology in order to place a large satellite system in orbit. For more on our efforts to conquer navigation, Prof. Fiete recommends Dava Sobel's book, *Longitude*.

Prof. Fiete then turned our attention to the brain, specifically, the hippocampus, which looks roughly like a sea horse (hence, its name). It is a well-studied part of the brain whose neural circuitry has largely been mapped out. Within the hippocampus, there are special cells called "place cells." In a mouse, these cells fire when the mouse is in a specific absolute location in a region, and there are place cells that cover the entire region. Because these cells fire consistently whenever the mouse is in a particular location in the region, independent of how the mouse came to that location or whatever it is looking at in that moment, we deduce that the place cells are not informed by visual cues. In contrast, there are "head direction cells," which fire when the head is looking in a particular compass direction. The physical proximity of these cells to each other corresponds to closeness in head direction, whereas with place cells, physically nearby place cells can respond to disparate locations in the region.

She also explained how there are cells that fire periodically as we walk in a fixed direction, but these cells fire with different periods. This enables them to have small periods, yet still inform us on how far we've gone for rather long distances via the Chinese Remainder Theorem.

Session 37 - Meet 6 October 16, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Clarise Han, Layla Jarrahy, Shauna Kwag, Yaqi Li, Hanna Mularczyk, Maya Robinson, Ella Wilson, Dora Woodruff

Take your favorite mathematical idea. How would you illustrate it to convey its mathematical content? If you come up with something that you're proud of, consider sharing it with us!

Session 37 - Meet 7 October 23, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Layla Jarrahy, Yaqi Li, Hanna Mularczyk, Maya Robinson

And if you come up with something visual that works better as a 3D object, consider realizing it with a 3D printer. If you don't know how to make a 3D printer print what you desire, teach yourself a language for describing objects to a 3D printer, such as OpenSCAD.

Session 37 - Meet 8 October 30, 2025	Mentors:	Elisabeth Bullock, Elsa Frankel, Clarise Han, Layla Jarrahy, Shauna Kwag, Yaqi Li, Hanna Mularczyk, Maya Robinson, Dora Woodruff

Can you think of a way to produce random positive integers in such a way that every positive integer has a nonzero probability of arising? In your scheme, what is the probability that n arises? What properties would any such method be required to adhere to? For example, if $p(n)$ is the probability that the positive integer n arises, then we would require that the sum of $p(n)$ over all positive integers be equal to 1. Are there any other properties that $p(n)$ must obey?

Calendar

Session 37: (all dates in 2025)

September	11	Start of the thirty-seventh session!
	19	
	25	
October	2	
	9	Ila Fiete, MIT
	16	
	23	
	30	
November	6	
	13	
	20	
	27	Thanksgiving - No meet
December	4	

Session 38: (all dates in 2026)

January	29	Start of the thirty-eighth session!
February	5	
	12	
	19	
	26	No meet
March	5	
	12	
	19	
	26	No meet
April	2	
	9	
	16	
	23	No meet
	30	
May	7	

Girls' Angle has run nearly 200 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____