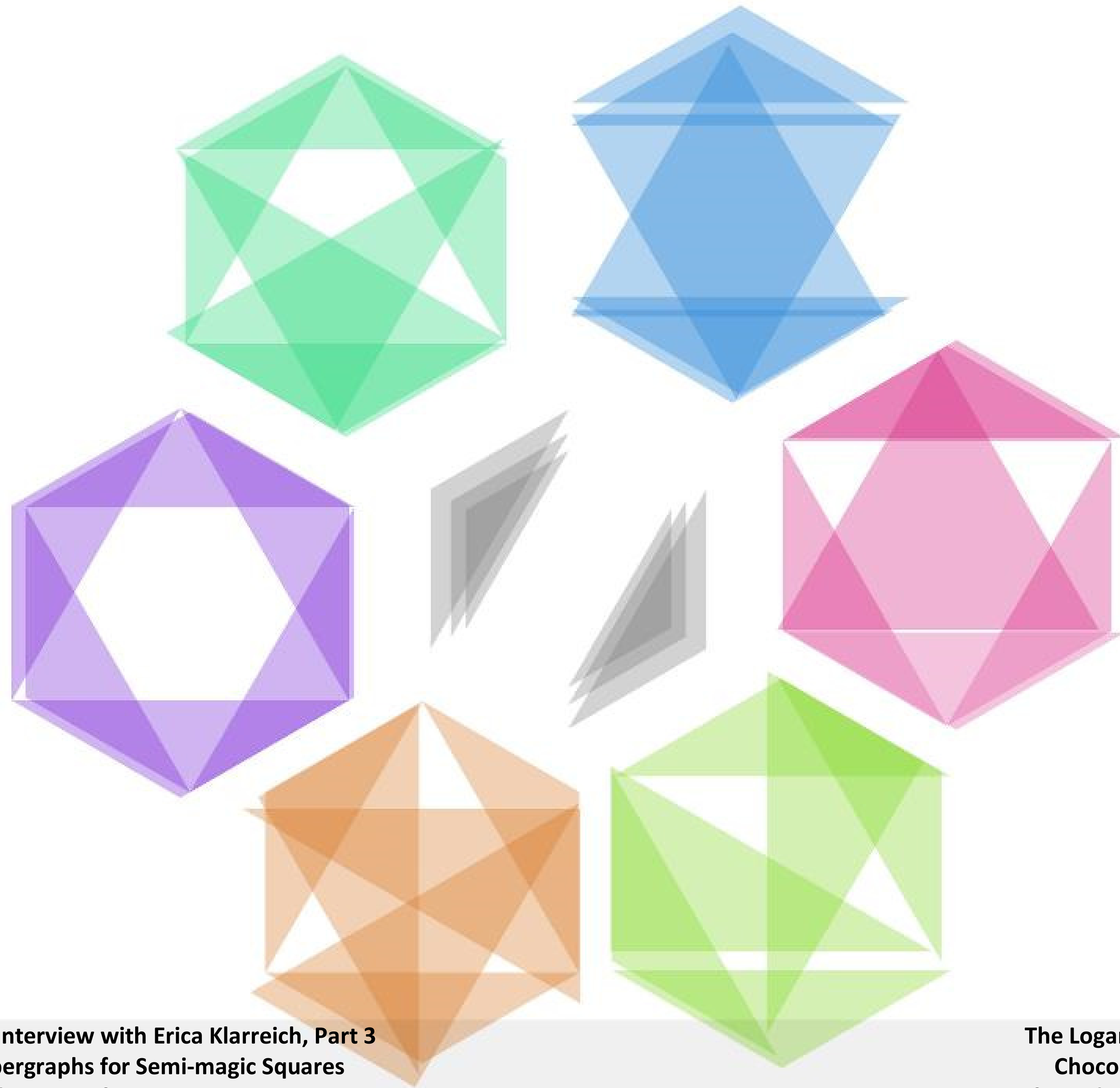
Girls' Angle Bulletin

April/May 2025 • Volume 18 • Number 4

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

In the last 18 years, many local math/science majors, grad students, and postdocs have worked as mentors at Girls' Angle. They're an astonishing group of women with diverse interests that go far beyond mathematics. We hope our members befriend our mentors. -Ken Fan, President and Founder

Girls' Angle Donors

A Heartfelt Thank You to our Donors!

Individuals

Uma Achutha
Amrita Ahuja
Nancy Blachman and David desJardins, founders of the Julia Robinson Mathematics Festival, jrmf.org.
Michael Beckman
Ravi Boppana
Marla Capozzi
Yungyuan Chang
Suhas Daftuar
Patricia Davidson
Ingrid Daubechies
Anda Degeratu
Ellen Eischen
Glenn and Sara Ellison
John Engstrom
Lena Gan
Jacqueline Garrahan
Courtney Gibbons
Vanessa Gould
Rishi Gupta
Larry Guth
Andrea Hawksley
Scott Hilton
Delia Cheung Hom and Eugene Shih
David Kelly
Stephen Knight and Elizabeth Quattrocki Knight
Anna Ma
Sarah Manchester
Brian and Darlene Matthews
Toshia McCabe
Alison and Catherine Miller
Mary O'Keefe
Beth O'Sullivan
Robert Penny and Elizabeth Tyler
Malcolm Quinn
Jeffrey and Eve Rittenberg
Marcine Snyder
Eugene Sorets
The Waldman and Romanelli Family
Marion Walter
Andrew Watson and Ritu Thamman
Brandy Wiegers
Brian Wilson and Annette Sassi
Lissa Winstanley
Cunyuan Zhang
The Zimmerman family
Anonymous

Nonprofit Organizations

Draper Laboratories
The Mathenaeum Foundation
Orlando Math Circle
Pro Mathematica Arte

Corporate Donors

Adobe
Akamai Technologies
Big George Ventures
D. E. Shaw
John Hancock
Maplesoft
Massachusetts Innovation & Technology Exchange (MITX)
MathWorks, Inc.
Microsoft
Microsoft Research
Nature America, Inc.
Oracle
Owl City Ventures, LLC

Girls' Angle Bulletin

The official magazine of Girls' Angle: A Math Club for Girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editors: Amanda Galtman
Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Hexagonal Hypergraphs*. Concept: Robert Donley. Execution: C. Kenneth Fan. Orbit representatives of 6×6 semi-magic squares with line sum 3. See page 7.

An Interview with Erica Klarreich, Part 3

…it's hard for me to really gauge how much of an impact my writing has in general. I would say that maybe the biggest source of gratification is when I hear from readers who enjoyed what I wrote.

Ken: I was going to ask you about that, if you had plans to write a larger format math novel even. But now – you've actually had thoughts about writing something not pertaining to math. What are you thinking about writing? Perhaps, a novel?

Erica: Yes. I told you how my sisters and I wrote these continuing stories when we were kids. And actually, as adults, my younger sister and I wrote the manuscript for a kid's book, for maybe 11- or 12-year-olds. We've been trying to get it placed, but it's a tough market. We've had to develop a very thick skin as we've been trying to find a home for this book.

Ken: I remember hearing Stephen King describe how when he was first trying to get a story published, he nailed his rejection letter to his bulletin board; he eventually got so many rejection letters that the nail fell out. So, he had to get another nail. That's his advice to young writers: get another nail.

Erica: Yeah.

Ken: I bet it's a wonderful story. I hope someone picks that up.

Erica: Well, I hope so, too, because we put a lot of work into it and I'm quite fond of it.

Ken: Does it have anything to do with math?

Erica: No, it does not have anything to do with math. When we were kids, one of the authors we loved is someone named Edward Eager who wrote in the 1950s. His most famous book is called *Half Magic*. He wrote these grounded fantasy stories about kids in this world who would discover some kind of magic and have adventures. He was our inspiration, and we wrote a story about some kids who are spending the summer in a town that has a Shakespeare festival. They discover this magical jester's staff and start having adventures in Shakespeare plays. We had a lot of fun writing it, and I will feel sad if it never sees the light of day. But I've gradually been reconciling myself to the idea that that's its likely fate.

Ken: Gosh, it amazes me, because you're such a successful writer. You've published so much. Don't they take that into account?

Erica: Well, I think the kind of writing I do does not have a lot of currency in the children's literature world.

Ken: I don't understand why it wouldn't. You have to write about imaginative ideas, and you convey them so well. And you tell stories in your writing.

Erica: I think that all the agents should be beating a path to my door to pick up this story, but it hasn't happened so far.

[Laughter.]

Ken: I hope it does happen. In the other direction, do you still do math?

Erica: No, I haven't done mathematics research in ages.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

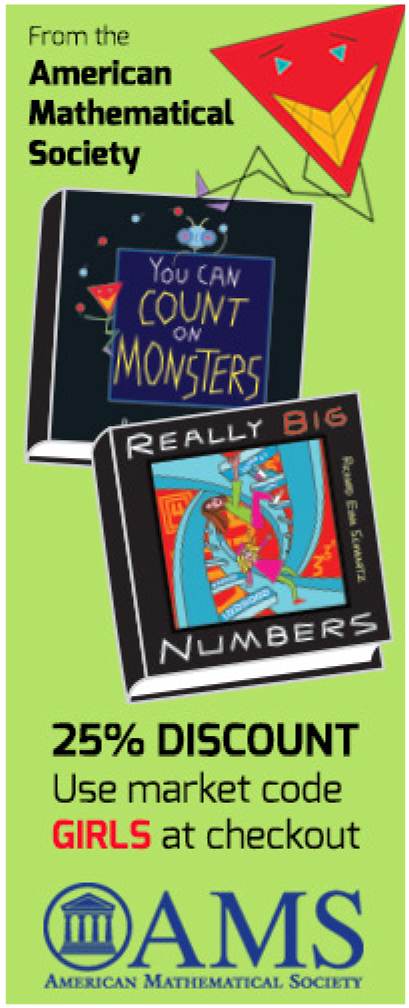
For this issue, those who do not subscribe to the print version will be missing out on a portion of the content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

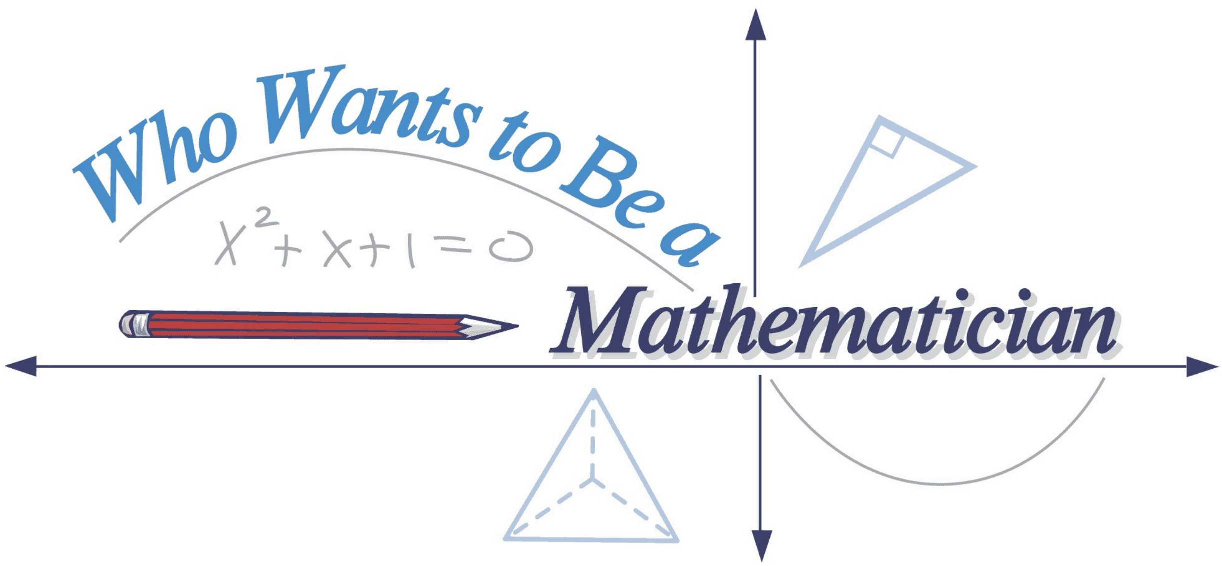
Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

Content Removed from Electronic Version



The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

Content Removed from Electronic Version



America's Greatest Math Game: Who Wants to Be a Mathematician.

Hypergraphs for Semi-magic Squares[1]

by Robert Donley[2]
edited by Amanda Galtman

We continue our investigation into semi-magic squares from the previous installment. In this part, we focus on general methods for counting semi-magic squares, postponing further development of regular edge colorings until the next installment. We introduce the use of hypergraphs for counting symmetries of matrices. We also use a conjecture by H. Anand, V. C. Dumir, and H. Gupta, proved by R. Stanley in the 1970s, for counting semi-magic squares with a given line sum.

It will be helpful to review some prior material: the previous installment; the orbit-stabilizer theorem in "Permutations and basic group theory: Part 2" (Volume 17, Number 5); and "Fibonacci Numbers and Multiset Counting" (Volume 15, Number 6) for generating functions, discrete convolution, and binomial series.

For applications to regular edge colorings, our interest lies in orbits of 0/1-semi-magic squares, which we consider first. For the Anand-Dumir-Gupta conjecture, we will need orbits of general semi-magic squares.

Definition: The **orbit** of the semi-magic square M is the set of all semi-magic squares obtained by repeatedly permuting the rows and columns of M.

Definition: Let J_n be the square matrix of size n with all entries equal to 1.

Then J_n is a semi-magic square with line sum n. The orbit of J_n contains only J_n itself.

Definition: If M is a 0/1-semi-magic square of size n, then the **complement** of M is $J_n - M$, denoted by M'.

Exercise: Prove that $(M')' = M$. Prove that if M has line sum k then M' is a 0/1-semi-magic square with line sum $n - k$. Prove that the orbit of M' consists of all semi-magic squares of the form $J_n - N$, where N is in the orbit of M.

Example: For 0/1-semi-magic squares of size 3, the orbits are determined by line sum, with a set of representatives given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Exercise: Calculate the orbit for each square above, and verify that the number of 0/1-semi-magic squares of size 3 is 14, as noted in the previous installment. For each M, calculate its complement M', and verify the preceding exercise on orbits and line sums.

---

[1] This installment is 20th in a series that began in Volume 15, Number 3. It is also part 7 of a subseries that began in Volume 17, Number 4.

[2]This content is supported in part by a grant from MathWorks.

Since $(M')' = M$, the number of 0/1-semi-magic squares of size n with line sum k equals the number with line sum $n - k$. Additionally, the set of semi-magic squares with line sum k decomposes into orbits under the action of row and column permutations. Thus, to count the number of 0/1-semi-magic squares with line sum k, it is enough to consider the orbit sizes, and for this we use the orbit-stabilizer theorem.

First, we describe the group of symmetries. In what follows, we need only enough detail of the group to calculate stabilizers.

Theorem: The group G generated by all permutations of rows and columns on the set of semi-magic squares of size n has $(n!)^2$ elements. Each element of G has the form $R(\sigma)C(\tau)$, where σ and τ are permutations of size n, and R and C denote row and column permutations, respectively.

Exercise: Prove that $C(\tau)R(\sigma) = R(\sigma)C(\tau)$. That is, whether we follow a row permutation by a column permutation or perform the column operation first, we get the same answer.

From the orbit-stabilizer theorem, the size of the orbit of M is given by $\left|O_M\right| = \dfrac{(n!)^2}{|\text{Stab}_G(M)|}$.

Example: For the semi-magic squares of size 3 with line sum 1, we know the orbit of the identity matrix I is the set of permutation matrices of size 3, so $|O_I| = 6$ and $|\text{Stab}_G(I)| = 6$. Note that there are six ways to permute the three 1s on the diagonal.

Exercise: For the six permutations σ of size 3, prove directly that $R(\sigma)C(\sigma)$ fixes I. Show that $R(\sigma)C(\sigma)M = P_\sigma M P_\sigma^{-1}$, where P_σ is defined by $P_\sigma(e_i) = e_{\sigma(i)}$, where e_i is the vector whose ith component is 1 and all other components are 0.

With the preceding arguments, we have calculated orbits and sizes in four cases for general n with line sum L. If $|L = k|$ denotes the number of 0/1-semi-magic squares with line sum k, then

$$|L = 0| = |L = n| = 1,\ |L = 1| = |L = n - 1| = n!.$$

Now consider the case of $|L = 2|$. Rather than work directly with M, we instead introduce the notion of a vertex-edge incidence matrix, which in turn allows us to define hypergraphs.

Definition: Let H be a graph with v vertices and e edges. The **vertex-edge incidence matrix** M_H of H is a matrix of size $v \times e$ with entries 0 or 1. We assign a column to each edge and a row to each vertex. In each column, we place values of 1 in the rows for the vertices of the edge. Otherwise, the entries equal zero.

The line sum of each column equals 2, and the line sum of each row equals the degree of the vertex. Thus, if each row has line sum 2 then every vertex of H has degree 2, and H is a disjoint union of cycles.

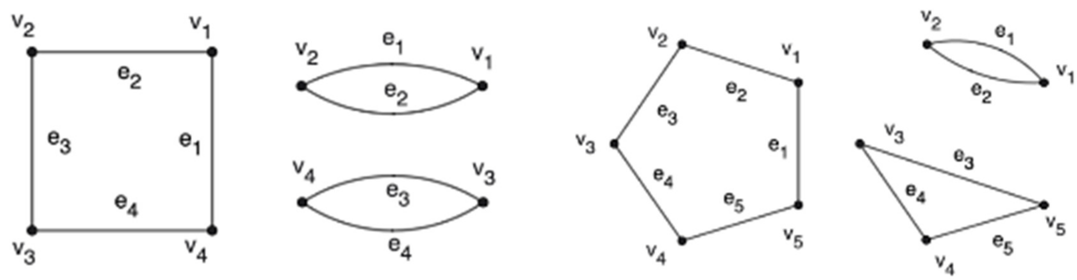
Note that permuting the rows or columns of M_H simply relabels the vertices or edges of H, potentially reorienting the graph. To be in the stabilizer of M_H, the permutation must preserve positions in M_H with value 1, so the orientation of the graph itself must be unchanged.

Using these observations, we can describe the semi-magic squares of size n with line sum 2:

- Each orbit is represented by an unlabeled graph H consisting of cycles, where we consider double edges as 2-cycles.
- The size of the stabilizer of M_H is given as a product: for each m-cycle, include a factor of $2m$, and, if a cycle of a given size occurs s times, multiply by a factor of $s!$.
- The size of each orbit is $(n!)^2/|\text{Stab}|$, and $|L = 2|$ is the sum of the orbit sizes.

Exercise: Using the Matching Rule, prove the above formula for the size of the stabilizer of the unlabeled graph. Compare with the final section of Volume 17, Number 4.

Examples: Consider the cases of $n = 4$ and $n = 5$ below. Each case admits a pair of cycle graphs as follows:



The corresponding incidence matrices M_H for each graph shown are the semi-magic squares

$$\begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\1&0&0&1\end{bmatrix}, \begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix}, \begin{bmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\1&0&0&0&1\end{bmatrix}, \begin{bmatrix}1&1&0&0&0\\1&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&1&0&1\end{bmatrix}$$

Let's calculate one orbit size for each case. For the pair of double edges, each 2-cycle contributes a factor of 4. Since there are two 2-cycles, we multiply by 2! to get a stabilizer of size 32, and $|O_M| = 24^2/32 = 18$. For $n = 5$, the stabilizer in the second case has size $4 \times 6 = 24$, so the corresponding orbit has size $120^2/24 = 600$.

Exercise: For $n = 4$, prove that $|L = 2| = 72 + 18 = 90$. Repeat for $n = 5$, in which case $|L = 2| = 1440 + 600 = 2040$.

We summarize our counts with a **rank sequence**, which is the list of the counts starting with $L = 0$. For $n = 3$, we have rank sequence (1, 6, 6, 1) with sum 14. For $n = 4$, we obtain the sequence (1, 24, 90, 24, 1) with sum 140. For $n = 5$, (1, 120, 2040, 2040, 120, 1) has sum 4322.

Exercise: When $n = 6$, find the four orbits for line sum 2. Then verify the following rank sequence for $n = 6$: (1, 720, 67950, X, 67950, 720, 1).

How do we find X in the previous exercise? That's not so easy, but only because counting symmetries is a matter of practice. To proceed, we extend the definition of the vertex-edge incidence graph in two ways. For $L = 3$, each column of the incidence matrix now contains three 1s, which requires an extension of the concept of a graph.

Definition: A **hypergraph** H consists of a non-empty set of vertices V and a set of hyperedges E. A **hyperedge** is a non-empty subset of vertices of any size.

Example: Every graph is a hypergraph such that every hyperedge has two vertices. If we allow loops, hyperedges can also consist of one vertex. It will be useful to allow repeated hyperedges.

Example: To solve for the number of 0/1-semi-magic squares of size n with $L = 3$, we have three 1s in each column of the vertex-edge incidence matrix. These hyperedges correspond to triangles. Thus, when $n = 6$, we are trying to solve the following hard exercise.

Exercise: Up to reorienting the vertices of the hexagon, find the number of ways to inscribe six triangles into a hexagon such that exactly three triangles meet at each vertex.

Partial answer: Seven, as seen on this issue's cover. The orbits sizes and their sum are

$$200 + 16200 + 21600 + 43200 + 43200 + 43200 + 129600 = 297200.$$

Thus, the total number of 0/1-semi-magic squares of size 6 is 434,542.

We'll omit the calculation of the stabilizers, but instead, to give some practice with symmetries of hypergraphs, we'll develop the related problem of finding **all** semi-magic squares with a given line sum L. We begin with $n = 3$, in which case the general form of the generating function for counting semi-magic squares of any size is given.

Denote the six permutation matrices of size three

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

by $P_1, \ldots, P_6$, respectively. As noted in the previous installment, these matrices satisfy the relation

$$J_3 = P_1 + P_2 + P_3 = P_4 + P_5 + P_6.$$

Suppose M is a semi-magic square of size 3. By the Birkhoff-von Neumann theorem, M equals a sum of permutation matrices. If m_0 denotes the smallest entry in M, then $M - m_0 J_3$ has smallest entry equal to 0.

Exercise: Prove that, if at least one entry of M equals 0, then M can be uniquely expressed as a sum of permutation matrices P_i. That is, $M = a_1P_1 + \ldots + a_6P_6$ for exactly one choice of non-negative integers a_i.

Exercise: Express each of the following semi-magic squares as a sum $M = M_0 + m_0J_3$, where m_0 is the smallest entry of M and at least one entry of M_0 equals 0. In each case, express M_0 as a sum of permutation matrices.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 6 \\ 6 & 3 & 4 \\ 4 & 6 & 3 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 4 \\ 4 & 4 & 5 \\ 3 & 6 & 4 \end{bmatrix}.$$

The decomposition in the previous exercise allows for a much simpler description of the underlying combinatorics for semi-magic squares of size 3. If M is written as a sum of permutation matrices as above, we assign to M the sextuple $(a_1, a_2, a_3, a_4, a_5, a_6)$. If all a_i are positive, this representation of M is not unique. For instance, by the relation, the sextuple

$$(a_1 + 1, a_2 + 1, a_3 + 1, a_4 - 1, a_5 - 1, a_6 - 1)$$

also represents M, and all sextuples that represent M are related in this manner.

Two useful facts arise from this ability to shift triples of 1s from right to left. First, for each M, there is a preferred sextuple for which at least one of a_4, a_5, or a_6 equals 0. Next, if m_0 is the smallest entry of M, then there are exactly $m_0 + 1$ ways to write M as a sum of permutation matrices.

Exercise: For each of the four matrices in the previous exercise, list all $m_0 + 1$ ways to express M as a sum of permutation matrices and as sextuples.

We now give an explicit formula for the number of semi-magic squares of size 3 with line sum k. In the following formulas, binomial coefficients are defined to vanish when the lower index exceeds the upper index.

Theorem: Let $H_3(k)$ be the number of semi-magic squares of size 3 with line sum k. Then

$$H_3(k) = \binom{k+5}{5} - \binom{k+2}{5}.$$

Proof: Consider the sextuple model. To obtain a semi-magic square with line sum k, we use multiset counting; that is, suppose we place k balls into 6 boxes, one for each a_i. The first term in the formula gives the number of these choices. To account for non-uniqueness, we must discard all sextuples in which all three of a_4, a_5, and a_6 are positive. For these choices, we place a single ball in each of the corresponding three boxes and then assign the remaining $k - 3$ balls in any manner to obtain the second term. □

If we apply the binomial series to the theorem, then the generating function for $H_3(k)$ is given by

$$S_3(x) = H_3(0) + H_3(1)x + H_3(2)x^2 + \ldots = \frac{1-x^3}{(1-x)^6}.$$

Exercise: Prove that the formula for $H_3(k)$ is a discrete convolution of the coefficients in $1 - x^3$ and the binomial series for $1/(1-x)^6$.

Exercise: Verify that $(1-x)(1+x+x^2) = 1-x^3$, so that

$$S_3(x) = \frac{1+x+x^2}{(1-x)^5}.$$

Repeat the method of the previous exercise to obtain the formula

$$H_3(k) = \binom{k+4}{4} + \binom{k+3}{4} + \binom{k+2}{4}.$$

Prove this formula directly by considering the ways in which at least one of a_4, a_5, and a_6 vanishes.

Exercise: Prove that these formulas agree by expanding the binomial coefficients into polynomials in k. For this polynomial, verify that $H_3(-1) = H_3(-2) = 0$. Calculate $H_3(k)$ for $0 \le k \le 5$.

Exercise: For $0 \le k \le 4$, find a representative for each orbit, determine the size of the orbit, and verify the counts from the previous exercise. After reading the next section, find the hypergraphs associated to each orbit.

For $n = 4$, we find that there are too many relations to make the previous technique work. Instead, we calculate $H_4(k)$ for small values using hypergraphs with weighted vertices and apply the Anand-Dumir-Gupta conjecture to find the general formula.

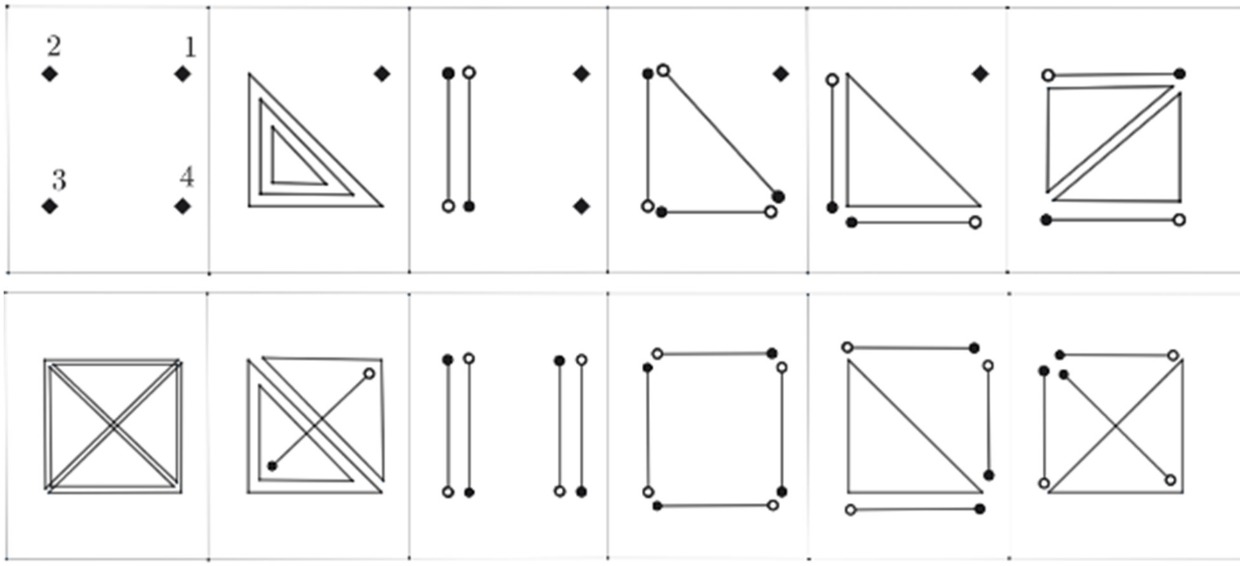
Note that $H_4(0) = 1$ and $H_4(1) = 24$. To calculate $H_4(2)$, first recall that we already saw that there are 90 0/1-semi-magic squares of size 4 with line sum 2. Since no entry can be greater than 2, there are only 3 more orbits, with representatives given by

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The weighted hypergraphs associated to these matrices are constructed as before, except that the hyperedge corresponding to a column with a 2 is just a vertex with weight 2. In effect, a proper edge is collapsed into a double vertex.

Exercise: Draw the hypergraphs for the matrices above, using a shaded circle for single vertices and open circles for vertices of weight 2. Calculate the stabilizers and the size of the corresponding orbits. Prove that there are no further orbits.

The previous exercise gives $H_4(2) = 18 + 24 + 72 + 72 + 96 = 282$. To calculate $H_4(3)$, we also consider vertices of weight 3, which we denote by a shaded diamond; edges with one vertex of weight 2; and nested triangles. The hypergraphs corresponding to this case are as follows:



For examples of calculations, the stabilizer in the second hypergraph above has size 36. Not only are the triangles preserved by permuting the vertices, but we obtain another factor of six by permuting the triangles among themselves. On the other hand, the stabilizer of the fourth hypergraph has size 3. We might guess the edges give six symmetries, but we cannot interchange shaded and unshaded vertices.

Exercise: List the corresponding vertex-edge incidence matrix for each hypergraph above. Verify the semi-magic property and line sum for each matrix and prove that no two of these matrices lie in the same orbit. Prove there are no other orbits.

Also compare symmetries of the hypergraphs with the corresponding permutations of rows and columns in M_H.

Exercise: The following list gives the size of the stabilizers for these hypergraphs:

1, 2, 2, 3, 4, 4, 4, 6, 8, 24, 24, 36.

Match each hypergraph with its stabilizer size.

Exercise: Use the orbit-stabilizer theorem to prove that $H_4(3) = 2008$.

With Stanley's proof of the Anand-Dumir-Gupta conjecture (top of next page), we now have enough data to compute the generating function and general formula for $H_4(k)$. The theorem states that the second form of the generating function for $H_3(k)$ holds in general with further restrictions. Explicit formulas for $H_n(k)$ are known up to the case of $n = 9$, which was calculated by M. Beck and D. Pixton in 2003 with the use of computers.

Theorem: The generating function for $H_n(k)$ is given by a rational function of the form

$$S_n(x) = H_n(0) + H_n(1)x + H_n(2)x^2 + \ldots = \frac{P_n(x)}{(1-x)^{(n-1)^2+1}},$$

where $P_n(x) = x^s + a_{s-1}x^{s-1} + \ldots + a_1x + a_0$. Furthermore, $s = n^2 - 3n + 2$ and $a_i = a_{s-i}$ for $0 \le i \le s$.

If we apply the binomial series and expand the binomial coefficients into polynomials, these conditions imply that $H_n(k)$ extends to a polynomial in k of degree $(n - 1)^2$ such that

- $H_n(-k) = (-1)^{(n-1)^2} H_n(k-n)$, and
- $H_n(-k) = 0$ for $1 \le k \le n-1$.

Stanley also conjectured and proved that each $a_i \ge 0$ and that $a_0 \le a_1 \le \ldots \le a_{\lceil \frac{s}{2} \rceil}$, where $\lceil x \rceil$ is the ceiling function (that is, the function that returns the smallest integer greater than or equal to x).

For the case of $n = 4$, we have

$$S_4(x) = H_4(0) + H_4(1)x + H_4(2)x^2 + \ldots = \frac{P_4(x)}{(1-x)^{10}},$$

where

$$P_4(x) = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_2x^2 + a_1x^1 + 1.$$

If we apply the binomial series with discrete convolution, we obtain the formula

$$H_4(k) = \binom{k+9}{9} + a_1\binom{k+8}{9} + a_2\binom{k+7}{9} + a_3\binom{k+6}{9} + a_2\binom{k+5}{9} + a_1\binom{k+4}{9} + \binom{k+3}{9}.$$

Exercise: Using the values for $H_4(k)$ ($1 \le k \le 3$), prove that $a_1 = 14$, $a_2 = 87$, and $a_3 = 148$. Calculate $H_4(4)$ and $H_4(5)$. See entry A001496 at the Online Encyclopedia of Integer Sequences.

Exercise: For $k = 4$, list the corresponding orbits, hypergraphs and stabilizers. In this case, there are five hyperedge types, including a tetrahedron or a quadruple vertex. First, determine all possible configurations at a single vertex, and then eliminate all cases with a tetrahedron or quadruple vertex. It will also help to transpose known cases with asymmetric incidence matrices. If you can program, a better approach is to generate the list and search for distinct orbit representatives.

Taylor Series for Tangent

by Ken Fan | edited by Jennifer Sidney

Emily: Did you notice that the teacher explained the Taylor series expansions for sin(x) and cos(x) at $x = 0$, but never mentioned the Taylor series for the tangent function?

Jasmine: Yes.

Emily: So of course that made me wonder what the Taylor series for tan(x) around $x = 0$ is.

Jasmine: Me too.

Emily: Do you want to figure it out now?

Jasmine: Sure!

Emily: Great! So I guess we need to figure out the nth order derivatives of tan(x).

Jasmine: For the first derivative, we know tan(x) = sin(x)/cos(x), so we can apply the quotient rule.

Emily and Jasmine compute the derivative.

Emily: I get that the derivative of tan(x), with respect to x, is $1/\cos^2(x)$.

Jasmine: Same here!

Emily: On to the second derivative!

Jasmine: I get $2\sin(x)/\cos^3(x)$.

Emily: Check!

Emily and Jasmine take a bit longer to compute the third derivative.

Jasmine: Did you get

$$\frac{d^3\tan(x)}{dx^3} = 2\frac{\cos^2(x)+3\sin^2(x)}{\cos^4(x)}?$$

Emily: Yes. These expressions are getting bigger and bigger.

Jasmine: I don't see any patterns. I guess we compute the fourth derivative?

Emily: Okay, I'll work it out. But can you spot me for errors?

Jasmine: Sure!

After fixing up several pesky errors, Emily and Jasmine produce the fourth derivative essentially like this (condensed from their scratch work, which occupies two sheets!):

$$\begin{aligned}
\frac{d^4 \tan(x)}{dx^4} &= \frac{d}{dx} 2\frac{\cos^2(x)+3\sin^2(x)}{\cos^4(x)} \\
&= 2\frac{\cos^4(x)\frac{d}{dx}(\cos^2(x)+3\sin^2(x)) - (\cos^2(x)+3\sin^2(x))\frac{d}{dx}\cos^4(x)}{\cos^8(x)} \\
&= 2\frac{\cos^4(x)(4\sin(x)\cos(x)) - (\cos^2(x)+3\sin^2(x))(-4\cos^3(x)\sin(x))}{\cos^8(x)} \\
&= 2\frac{8\sin(x)\cos^5(x)+12\cos^3(x)\sin^3(x)}{\cos^8(x)} \\
&= 2\frac{8\sin(x)\cos^2(x)+12\sin^3(x)}{\cos^5(x)}.
\end{aligned}$$

Emily: Whew! These expressions are getting longer and longer.

Jasmine: Maybe we should think about a common, uniform way to express these derivatives. If there are any patterns, hopefully that'll make them easier to spot.

Emily: Good idea! So far, the denominators have all been powers of cosine. To what extent can we express the numerators using only the cosine function?

Jasmine: It looks like the first and third derivatives have numerators that can be expressed entirely in terms of cosine, but the second and fourth derivatives look like $\sin(x)$ times an expression that only involves cosine. Let's make this explicit:

n	$\frac{d^n}{dx^n}\tan(x)$
1	$\frac{1}{\cos^2(x)}$
2	$2\frac{\sin(x)}{\cos^3(x)}$
3	$2\frac{3-2\cos^2(x)}{\cos^4(x)}$
4	$8\sin(x)\frac{3-\cos^2(x)}{\cos^5(x)}$

Emily: Hey! All the numerators can be expressed entirely in terms of sine. Let's try that:

n	$\frac{d^n}{dx^n}\tan(x)$
1	$\frac{1}{\cos^2(x)}$
2	$2\frac{\sin(x)}{\cos^3(x)}$
3	$2\frac{1+2\sin^2(x)}{\cos^4(x)}$
4	$8\frac{2\sin(x)+\sin^3(x)}{\cos^5(x)}$

Jasmine: I see that the denominators are increasing powers of cosine, and the numerators look like polynomials in sine, also with increasing degree.

Emily: If we insist that the coefficient of the highest power of sine be 1, then the constant factors in front of the fractions seem to be powers of 2; that's because in that third derivative, we'd have to factor out an additional factor of 2 from the numerator to make the coefficient of $\sin^2(x)$ be 1.

Jasmine: Nice … so let's hypothesize that the nth derivative of $\tan(x)$, with respect to x, has the form

$$2^{n-1}\frac{p_n(\sin(x))}{\cos^{n+1}(x)},$$

where $p_n(x)$ is a monic polynomial of degree $n - 1$.

Emily: Actually, I was thinking that the power of 2 in front is $n - 1$, which is the same as the degree we want for the polynomial $p_n(x)$. That makes me wonder if we should instead have the form be

$$\frac{p_n(2\sin(x))}{\cos^{n+1}(x)},$$

where $p_n(x)$ is a monic polynomial of degree $n - 1$. This seems to work for these first four derivatives; if we do it this way, we can drop the constant power of 2 in the front.

Jasmine: So $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = 2 + x^2$, and $p_4(x) = 8x + x^3$? Yes, I like that! Let's do that.

Emily: Let's try to prove that this form works for derivatives of every order for $\tan(x)$.

Jasmine: Okay. We've seen that this form works for the first four derivatives, so let's try to prove that the form works by induction on n. Assume it works for the nth derivative. Does it work for the $(n + 1)$-th derivative? In other words, does the derivative of the form have the same form?

Emily: Let's see!

$$\begin{aligned}\frac{d}{dx}\frac{p_n(2\sin(x))}{\cos^{n+1}(x)} &= \frac{\cos^{n+1}(x)\frac{d}{dx}p_n(2\sin(x)) - p_n(2\sin(x))\frac{d}{dx}\cos^{n+1}(x)}{\cos^{2n+2}(x)} \\ &= \frac{2\cos^{n+2}(x)p'_n(2\sin(x)) + (n+1)\sin(x)\cos^n(x)p_n(2\sin(x))}{\cos^{2n+2}(x)} \\ &= \frac{2\cos^2(x)p'_n(2\sin(x)) + (n+1)\sin(x)p_n(2\sin(x))}{\cos^{n+2}(x)} \\ &= \frac{2(1-\sin^2(x))p'_n(2\sin(x)) + (n+1)\sin(x)p_n(2\sin(x))}{\cos^{n+2}(x)}.\end{aligned}$$

Jasmine: The denominator is what we hoped for, and the numerator is a polynomial in $\sin(x)$, but we want the numerator to be a *monic* polynomial in $2\sin(x)$. We need to identify the coefficient of the highest power of $\sin(x)$ in that numerator.

Emily: Since $p_n(x) = x^{n-1}$ + (lower order terms), the derivative of $p_n(x)$ will have lead term $(n-1)x^{n-2}$. So the first term in the numerator contributes $-2\sin^2(x)(n-1)(2\sin(x))^{n-2}$ to the highest order term, and the second term contributes $(n+1)\sin(x)(2\sin(x))^{n-1}$. Combining these terms, we find that the highest order term (in $\sin(x)$) is

$$(-(n-1)2^{n-1} + (n+1)2^{n-1})\sin^n(x) = 2^n\sin^n(x) = (2\sin(x))^n.$$

Perfect!

Jasmine: It works!

Emily: Our calculation actually gives us a recursion formula for $p_n(x)$:

$$\begin{aligned}p_{n+1}(x) &= 2(1-x^2/4)p'_n(x) + (n+1)(x/2)p_n(x) \\ &= \frac{(n+1)xp_n(x) + (4-x^2)p'_n(x)}{2}.\end{aligned}$$

Jasmine: That's a complicated recursion formula! But I guess it does make it easier to compute the fifth derivative of $\tan(x)$. I get $p_5(x) = (5x(8x + x^3) + (4 - x^2)(8 + 3x^2))/2 = 16 + 22x^2 + x^4$, so

$$\frac{d^5\tan(x)}{dx^5} = \frac{16 + 88\sin^2(x) + 16\sin^4(x)}{\cos^6(x)}.$$

To be continued…

Hypercube Sections: Pascal Revisited

by Addie Summer | edited by Amanda Galtman

Here, Addie takes the n-dimensional hypercube to be the points in n-dimensional space with coordinates $(x_1, x_2, x_3, \ldots, x_n)$, where $-1 \le x_k \le 1$ for $k = 1, 2, 3, \ldots, n$. Its vertices are the points each of whose coordinates is either 1 or -1.

Let v be the vertex whose coordinates are all equal to 1. For each integer k from 0 to n, let $V_{k,n}$ be the set of vertices that have exactly k coordinates equal to -1, and let $C_{k,n}$ be the convex hull of the vertices in $V_{k,n}$. If it is clear what n is, Addie might omit the n and simply write V_k and C_k.

So far, we've figured out that $C_{2,n}$ is an $(n-1)$-dimensional hyperpolyhedron with ${}_nC_2$ vertices and $n(n-1)(n-2)^2/6$ equilateral triangular faces. Each vertex is a corner of $(n-2)^2$ triangular faces.

Should we next try to figure out what its 3D faces are? Or perhaps we should look at its highest-dimensional faces, which are $(n-2)$-dimensional?

Or, perhaps there's an entirely different way to approach understanding what the hyperpolyhedron $C_{2,n}$ looks like. If I try to find the k-dimensional faces for each k using the methods I've been using so far, I foresee a lot of algebra in my future!

Surely, there's a more geometric way of exploring these shapes?

To find a more geometric approach, I need to think more about the geometry of the situation. The original way of separating the vertices of the hypercube into the various sets $V_{k,n}$ was via their distance to v. It's neat that the vertices of a hypercube separate themselves into layers like that. But hypercubes have a lot of other structure, too. For one thing, they're all regular prisms. The n-dimensional hypercube is a regular prism whose bases are $(n-1)$-dimensional hypercubes.

Viewing it as a prism, we can take the bases to be the intersection of our hypercube with the hyperplanes $x_1 = 1$ and $x_1 = -1$, for example. These two bases split the set of vertices of the hypercube in half.

What are the implications of this prism structure on the geometry of $C_{2,n}$?

Let's see. One implication is that the vertices of $C_{2,n}$, that is, the set $V_{2,n}$, also splits into two subsets: vertices whose first coordinate is +1 and those whose first coordinate is -1. The vertices in $V_{2,n}$ for which $x_1 = 1$ are those for which exactly two coordinates, other than the first, have the value -1…

Wait a sec…that's just like $V_{2,n-1}$! That is, the intersection of $C_{2,n}$ with the hyperplane $x_1 = 1$ is congruent to $C_{2,n-1}$. The isometry that shows this sends the point $(1, y_1, y_2, y_3, \ldots, y_{n-1})$ in n-dimensional space to the point $(y_1, y_2, y_3, \ldots, y_{n-1})$ in $(n-1)$-dimensional space. The vertices in $V_{2,n}$ whose first coordinate is 1 are sent exactly to the vertices of $V_{2,n-1}$ by this isometry. And the vertices in $V_{2,n}$ that are contained in the hyperplane $x_1 = -1$ have the same structure as the vertices in $V_{1,n-1}$, by similar reasoning.

In other words, two of the $(n-2)$-dimensional faces of $C_{2,n}$ are congruent to $C_{1,n-1}$ and $C_{2,n-1}$. These faces are parallel to each other and sandwich the rest of $C_{2,n}$. Because every vertex of $C_{2,n}$ is captured by one of these two $(n-2)$-dimensional faces, $C_{2,n}$ (which is the convex hull of the vertices in $V_{2,n}$) can be built by taking the union of all line segments that connect points of one base to points of the other! In other words, $C_{2,n}$ is like a prism, except it has different bases on top and bottom. One base looks like $C_{1,n-1}$, while the other looks like $C_{2,n-1}$.

When $n = 3$, we saw that $C_{2,3}$ is an equilateral triangle. But this new perspective tells us that we can see the triangle as a line segment ($C_{1,2}$) and a point ($C_{2,2}$), together with every line segment that connects points of one with the other. And when $n = 4$, this latest perspective tells us that we can view the octahedron ($C_{2,4}$) as two equilateral triangles ($C_{1,3}$ and $C_{2,3}$) placed parallel to each other (and rotated), together with every line segment connecting a point of one triangle with a point of the other.

The value of viewing things in this way is that it generalizes to all $C_{2,n}$!

In this way, we can understand $C_{2,n}$ inductively. Starting with a point ($C_{2,2}$), we place a line segment ($C_{1,2}$) somewhere outside the point and draw all the segments connecting the line segment to the point, to obtain an equilateral triangle ($C_{2,3}$). We then place an equilateral triangle ($C_{1,3}$) somewhere outside and parallel to our equilateral triangle ($C_{2,3}$) and draw all the segments connecting points of one with the other, to obtain a regular octahedron ($C_{2,4}$). We then place a regular tetrahedron ($C_{1,4}$) somewhere outside and parallel to our regular octahedron ($C_{2,4}$) (in 4D space) and draw all the segments connecting points of one with the other, to obtain $C_{2,5}$, and so on.

The construction is analogous to the way the sequence of n-dimensional simplices or n-dimensional hypercubes can be constructed iteratively, starting with a point. For simplices, at each stage, we place a new point (in a higher dimension) and connect this new point to all the points of the previous iteration. For hypercubes, instead of placing a new point at each step, we place a copy of the latest iteration (in a higher dimension). For $C_{2,n}$, instead of placing a point or a copy of the latest iteration, we place an $(n-2)$-dimensional simplex! Wild!

In these inductive descriptions, I was imprecise about the relative positions of the two bases. Actually, the construction requires that the relative positions of the two bases correspond exactly to the way they exist as cross sections of $C_{2,n}$ with the parallel hyperplanes $x_1 = 1$ and $x_1 = -1$.

Actually, can't we play this "prism" game with $C_{k,n}$ for any k, not just for $C_{2,n}$?

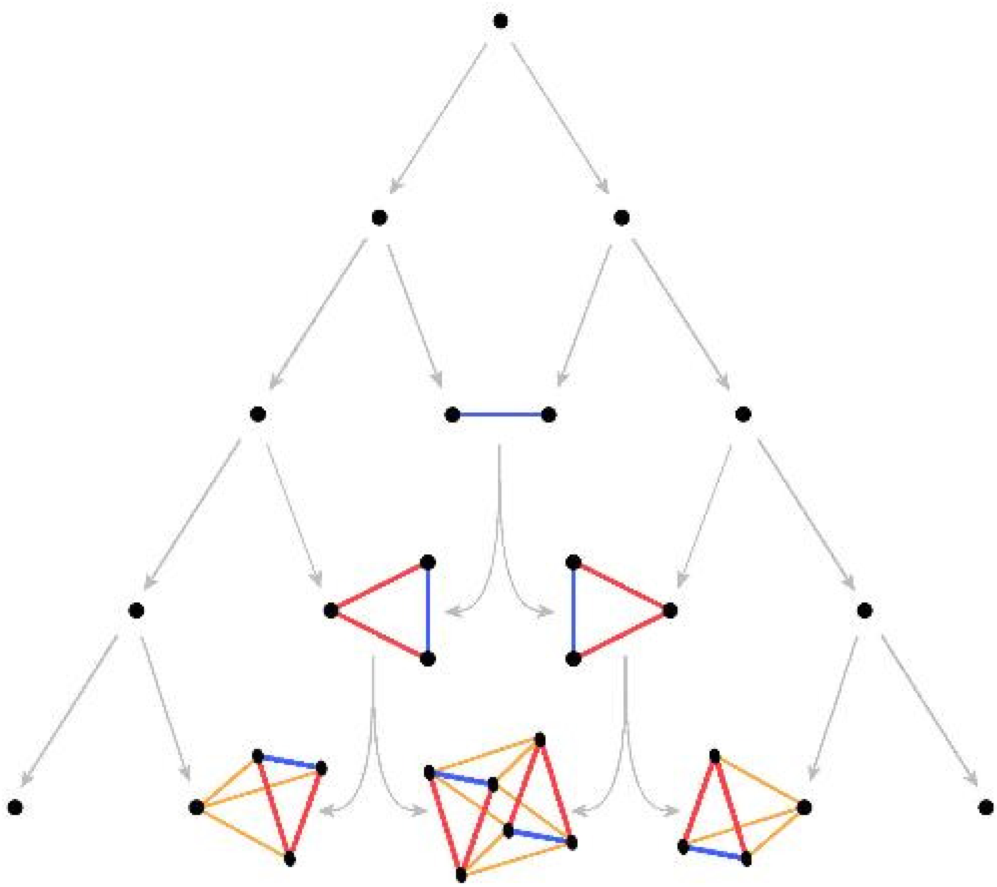
After all, the same reasoning shows that the cross section of $C_{k,n}$ by the hyperplane $x_1 = 1$ is congruent to $C_{k,n-1}$, and the cross section of $C_{k,n}$ by the hyperplane $x_1 = -1$ is congruent to $C_{k-1,n-1}$. In this way, we can see that $C_{k,n}$ is a "prism" [1] with the two different bases $C_{k,n-1}$ and $C_{k-1,n-1}$.

Isn't it neat how this structure mirrors the way each entry in Pascal's triangle is the sum of the two entries just above it? In other words, we can make a new version of Pascal's triangle where each entry is a geometric shape instead of a number. All the 1's in Pascal's triangle are replaced with points. Other entries are replaced by the "prism" formed by taking the two shapes just above and placing them in parallel hyperplanes and drawing all the line segments connecting

[1] A "prism" with two different bases B_1 and B_2 is known as the **join** of B_1 and B_2. It is sometimes denoted by $B_1 \ast B_2$. More precisely, let B_1 and B_2 be two disjoint shapes and suppose that any two line segments that connect a point of B_1 with a point of B_2 are either disjoint or intersect at their endpoints. Then the join of B_1 and B_2 consists of B_1 and B_2 and all the line segments that connect a point of B_1 with a point of B_2.

points of one shape to the other! The original Pascal's triangle gives the number of vertices in the corresponding shape in our geometric Pascal's triangle.

This is so beautiful that even though I am trying to avoid drawings, I just had to sketch this idea:



Geometric Pascal's Triangle

Each shape is the "prism" formed by using the two shapes directly above as bases. The nth row gives snapshots of cross sections of the n-dimensional hypercube by an $(n - 1)$-dimensional hyperplane passing through. The number of vertices of each shape is given by Pascal's triangle.

There are many, many identities involving entries of Pascals' triangle (a.k.a. binomial coefficients), such as the "hockey stick pattern." How many such identities can you reinterpret geometrically by using this geometric version of Pascal's triangle?

Let's see if thinking about $C_{k,n}$ in this way enables us to better understand its various faces. Let's assume that $n > 3$, because we understand the cases $n = 1$, 2, and 3. And let's continue to assume (from the last installment) that $0 < k < n$, so that we don't have to worry about the points on the edges of the geometric Pascal's triangle.

What we've found is that $C_{k,n}$ is a "prism" formed by taking the union of all line segments joining points of copies of $C_{k-1,n-1}$ and $C_{k,n-1}$, where our copy of $C_{k-1,n-1}$ lives in the hyperplane $x_1 = -1$ and our copy of $C_{k,n-1}$ lives in the hyperplane $x_1 = 1$. Specifically, our copy of $C_{k-1,n-1}$ is the image of $C_{k-1,n-1}$ under the map that sends the point $(y_2, y_3, y_4, \ldots, y_n)$ to the point $(-1, y_2, y_3, y_4, \ldots, y_n)$, and our copy of $C_{k,n-1}$ is the image of $C_{k,n-1}$ under the map that sends the point $(y_2, y_3, y_4, \ldots, y_n)$ to the point $(1, y_2, y_3, y_4, \ldots, y_n)$ in $C_{k,n}$.

Every d-dimensional face in one of the bases is a d-dimensional face of $C_{k,n}$. But we also have d-dimensional faces on the "sides" of the "prism." How can we identify those?

If we look for the one-dimensional edges, for example, we already know that joining a vertex in one base to a vertex in the other base does not necessarily create an edge. This first happens in the regular octahedron $C_{2,4}$, where the two bases are triangles. Each vertex in one triangle can be joined to two vertices in the other to form edges, but the line segment to the third vertex passes through the interior of the octahedron.

Instead of trying to figure out what the lateral surface of our "prism" consists of, perhaps we can exploit more of the symmetry of the hypercube to help us find *all* the faces.

After all, our geometric Pascal's triangle was a consequence of looking at the intersection of $C_{k,n}$ with the hyperplanes $x_1 = 1$ and $x_1 = -1$, but there's no special reason to use these two hyperplanes that are defined by fixing the first coordinate…we could just as well have used the hyperplanes $x_2 = 1$ and $x_2 = -1$, or $x_i = 1$ and $x_i = -1$ for any integer i between 1 and n, inclusive. By symmetry, we would have read the same story. Each such hyperplane intersects $C_{k,n}$ in one of its $(n-2)$-dimensional faces. So several $(n-2)$-dimensional faces just "fall into our lap." Let's embrace this gift and analyze these faces further.

Since there are n coordinates and each can be set to either 1 or -1, we obtain a total of $2n$ $(n-2)$-dimensional faces. Half of them are congruent to $C_{k,n-1}$ (namely, the ones where a coordinate is fixed to 1) and the other half are congruent to $C_{k-1,n-1}$ (the ones where a coordinate is fixed to -1). Well, actually, if $k = 1$ or $k = n-1$, we get only n $(n-2)$-dimensional faces, because half of the hyperplanes intersect $C_{k,n}$ in just a single point. But when $k = 1$ or $k = n-1$, we know that $C_{k,n}$ is a regular simplex. So let's assume that $1 < k < n-1$, so that we're always in the situation where these $2n$ hyperplanes give us $2n$ $(n-2)$-dimensional faces.

Could these $2n$ faces be *all* of the $(n-2)$-dimensional faces of $C_{k,n}$?

What, exactly, are we doing? We are looking at the convex hulls of subsets of the vertices of an n-dimensional hypercube, namely those vertices contained in the hyperplane

$$x_1 + x_2 + x_3 + \ldots + x_n = n - 2k.$$

Might $C_{k,n}$ *be* the cross section of the hypercube by the hyperplane? If it were, then it would be easier to know what the $(n-2)$-dimensional faces of $C_{k,n}$ are. But the way we defined $C_{k,n}$, we know only that the cross section *contains* $C_{k,n}$. That's because both the hypercube and the hyperplane are convex, so the cross section is convex, and since it contains all the vertices in $V_{k,n}$, it must contain $C_{k,n}$. But are they actually *one and the same*?

If they were the same, then they would have the same set of vertices. So if we can show that the vertices of the cross section are exactly the points in $V_{k,n}$, that would show that $C_{k,n}$ coincides with the cross section. What are the vertices of the cross section? Any such vertex would have to be on the boundary of the hypercube. And since a k-dimensional face of a hypercube is a piece of a k-dimensional hyperplane, if our cross sectioning hyperplane $x_1 + x_2 + x_3 + \ldots + x_n = n - 2k$ intersects a two- or higher-dimensional face at a point in its interior, that point cannot be a vertex. So, the vertices are intersections of our cross sectioning hyperplane with edges of the hypercube. An edge of our hypercube corresponds to setting each of $n - 1$ coordinates independently to either 1 or -1, while allowing the remaining coordinate to range over the values from -1 to 1, inclusive. Suppose i is the free coordinate so that the edge consists of the points $(x_1, x_2, x_3, \ldots, x_n)$ where $-1 \le x_i \le 1$ and $x_j = \pm 1$ for $j \ne i$. Let $p = (p_1, p_2, p_3, \ldots, p_n)$ be a point in the intersection of this edge with our cross sectioning hyperplane, so that p also satisfies the equation of the hyperplane:

$$p_1 + p_2 + p_3 + \ldots + p_n = n - 2k.$$

Isolating p_i, we find that p_i must be an integer. In fact, p_i must be an odd integer. That's because adding up a bunch of 1s and -1s has the same parity as the number of numbers being added. In this case, p_i has the same parity as $n - 2k - (n - 1) = 1 - 2k$. Since $-1 \le p_i \le 1$, p_i must either be 1 or -1, or have no solution at all (which happens when the hyperplane does not intersect the edge). So if there is an intersection, all of p's coordinates are +1 or -1, which means that p is in $V_{k,n}$, as we hoped!

So, $C_{k,n}$ *is* the cross section of our hypercube with the hyperplane $x_1 + x_2 + x_3 + \ldots + x_n = n - 2k$.

And that means that the boundary of $C_{k,n}$ *is* the intersection of the hyperplane with the boundary of the hypercube, and so its $(n - 2)$-dimensional faces are, indeed, the intersections of $C_{k,n}$ with the $2n$ hyperplanes $x_i = \pm 1$. (Recall that we are assuming that $1 < k < n - 1$, so that these intersections are never isolated points.)

In fact, we can find the d-dimensional faces of $C_{k,n}$ by looking at the intersection of the cross sectioning plane with the various $(d + 1)$-dimensional faces of the hypercube, which are all, themselves, $(d + 1)$-dimensional hypercubes. The $(d + 1)$-dimensional faces of our n-dimensional hypercube correspond to fixing $n - (d + 1)$ of the coordinates independently to either 1 or -1, and allowing the other coordinates to range freely over the values from -1 to 1, inclusive. Let's call the coordinates that are fixed to be either 1 or -1 the **fixed** coordinates, and let's call the coordinates that are allowed to range freely between -1 and 1, inclusive, the **free** coordinates. Let I be the set of free coordinates and suppose that the sum of the values of the fixed coordinates is S. Note that S has the same parity as $n - (d + 1)$. The intersection of this $(d + 1)$-dimensional face with the cross sectioning hyperplane corresponds to solutions of the equation

$$x_1 + x_2 + x_3 + \ldots + x_n = n - 2k$$

subject to these additional constraints: $-1 \le x_i \le 1$ for i in I, and $x_i = \pm 1$ for i not in I, where the specific value depends on i. Our equation can be written

$$\sum_{i \in I} x_i = n - 2k - S\,.$$

Notice that this linear equation can exactly be interpreted as the intersection of a hyperplane of the type we are studying with a $(d + 1)$-dimensional hypercube (in the space spanned by the $d + 1$ coordinates indexed by I). In other words, every d-dimensional face of $C_{k,n}$ is congruent to some cross section $C_{j,d+1}$ for some $0 < j < d + 1$! Another way to put it is that as we build our geometric Pascal's triangle row by row, each new object has facets that are congruent to shapes that appear in the rows above.

Armed with all this understanding, let's find the edges of $C_{k,n}$.

Since the only 1D-shape that will ever appear in our geometric Pascal's triangle is a line segment of length $\sqrt{8}$, *every edge of every* $C_{k,n}$ is a line segment of length $\sqrt{8}$!

Is the converse true? If two vertices in $V_{k,n}$ are $\sqrt{8}$ apart, then they must differ in only two coordinates, say, i and j. In each of the two vertices, the ith and jth coordinate values must have opposite sign, or else the vertices wouldn't both have coordinate sums equal to $n - 2k$. But that means the two vertices sit diagonally across from each other in the square facet of the hypercube where coordinates i and j are free and all others are fixed. Therefore, every pair of vertices that are $\sqrt{8}$ apart are, indeed, the endpoints of an edge of $C_{k,n}$.

For each vertex in $C_{k,n}$, k of its coordinates are equal to -1 and $n - k$ coordinates are equal to 1. This gives us $k(n - k)$ pairs of coordinates where one coordinate is 1 and the other is -1. Therefore, each vertex in $V_{k,n}$ is the endpoint of $k(n - k)$ edges. Since there are ${}_nC_k$ vertices in $V_{k,n}$ and each edge has two endpoints, the total number of edges must be

$$\frac{1}{2}k(n-k)\binom{n}{k} = \frac{n!}{2(k-1)!(n-k-1)!}.$$

Great! On to 2D faces.

Since the only 2D shapes in our geometric Pascal's triangle are both equilateral triangles ($C_{1,3}$ and $C_{2,3}$), *every 2D face of every* $C_{k,n}$ must be an equilateral triangle!

Let p be in $V_{k,n}$. How many triangular faces is p a vertex of? Every triangular face is the intersection of $C_{k,n}$ with a cubic face of the hypercube. Among the three free coordinates that define a cubic face, the intersection with $V_{k,n}$ must have either two 1s and one -1 or two -1s and one 1. So the number of triangular faces that have p as a vertex is the number of triples of coordinates of p that are neither all 1 nor all -1. Since we assume that $1 < k < n - 1$, there are always at least two coordinates that are 1 and at least two coordinates that are -1. Therefore, p is a vertex of

$${}_{n-k}C_2 \cdot k + (n-k) \cdot {}_kC_2 = (n-2)(n-k)k/2$$

triangular faces, and the total number of 2D faces is $\frac{1}{3}\frac{(n-2)(n-k)k}{2}\binom{n}{k} = \frac{n!(n-2)}{6(k-1)!(n-k-1)!}$.

To be continued…

The Logarithm

by Girls' Angle Staff

Let's understand the logarithm.

A disease is spreading through the population. Every week, the number of people who have been infected doubles. Right now, 1 person has been infected. In how many weeks will 8 people have been infected?

Since the number of people who have been infected doubles every week, there will be 2 such people in one week, then 4 after two weeks, and then 8 after three. So the answer is 3. Each week, we multiply the number of such people by 2, so we can say that after w weeks, the number of people who have been infected will be 2^w. In this way, we have solved the equation $2^w = 8$.

But we might want to know how many weeks it will take before n people have been infected for different values of n other than 8. That is, we might wish to solve the equation $2^w = n$. For example, the population of the United States is about 340,000,000 and we might want to know how many weeks it will take before everybody in the United States has been infected. That is, for what w is $2^w = 340,000,000$?

If you start at 1 and keep doubling, you'll discover that 2^{28} is less than 340,000,000, but 2^{29} is greater. So the answer is somewhere between 28 and 29 weeks. It wouldn't be accurate to say 28 weeks or 29 weeks. More accurately, it'll be about 28.34 weeks, but that's not exactly right either. It's a situation similar to finding the positive square root of 2. We know what we are looking for, but we cannot write down the exact answer in decimal notation. We can only write down an approximation. So rather than settle for an approximation of this square root, we instead invent notation and write $\sqrt{2}$. For the solution to $2^w = 340,000,000$, we do the same. We invent notation that tells us exactly that we mean the number that you have raise 2 to in order to get 340,000,000. That notation looks like this:

$$\log_2 340,000,000.$$

In this way, we can say that the value of w that satisfies the equation $2^w = n$ is $\log_2 n$.

In general, $\log_b n$ is the value to which b must be raised to produce n. Thus, $\log_3 81 = 4$, $\log_{10} 1,000 = 3$, and $\log_4 1024 = 5$. The function $\log_b$ is the "logarithm base b," or "the log base b" for short.

A common situation where we might use a logarithm to an unusual base is with bank accounts. A typical bank account might accrue interest at the rate of 2% per year. How long will it take for the money in such an account to double? The answer is $\log_{1.02} 2$ years.

But you might want to know a good decimal approximation to a logarithm such as $\log_{1.02} 2$. For that, the basic idea is as indicate above, when we figured out $\log_2 8$. But there are ways to speed up the computation of an approximation by finding properties of the logarithm and exploiting them to good advantage. We'll leave that up to you. What properties of the logarithm can you find and how would you use them to compute decimal approximations? See if you show that $\log_{1.02} 2$ is just over 35.

ChocoMath

by Lightning Factorial | edited by Jennifer Sidney

You love chocolate! Who doesn't?

But you just learned in health class that sugar is unhealthy, and your favorite chocolate has 5 grams of sugar per tablespoon!

You decide to do an experiment to find out the least amount of sugar you need for a piece of chocolate to taste good to you. To carry out this experiment, you purchase cocoa butter, cocoa powder, sugar, and a chocolate heart mold consisting of a dozen tablespoon-sized cavities

To make chocolate, you need to melt cocoa butter at a not-too-high temperature (perhaps in a pot held over the steam of boiling water), mix in a little bit of cocoa powder and sugar, then pour the mixture into a mold. The mold is set aside (or placed in a refrigerator) to allow the liquid to gel. That's it!

For your experiment, however, you wish to make a series of chocolates with different amounts of sugar. You decide to make chocolates containing sugar in amounts ranging from no sugar at all to 5 grams, in increments of half a gram. Since there are 11 different amounts of sugar, you need to make 11 tablespoons of chocolate mixture.

Here are some facts about the ingredients:

- 100 grams of cocoa butter is 7 tablespoons
- 100 grams of cocoa powder is 16 tablespoons

1. You have a scale that gives weight measurements in grams. You want to start by making a chocolate mixture consisting of 25% cocoa powder and 75% cocoa butter by weight. How many grams of cocoa butter and cocoa powder do you need to produce a mixture totaling 11 tablespoons? (You're not concerned about how adding sugar will increase the volume.)

2. What is the total amount of sugar you need, in grams?

Mixing sugar directly into the tablespoon-sized cavities of the mold is difficult, because you have to stir vigorously to get the sugar dissolved and evenly spread out, and that risks spillage (a tragic loss of chocolate!). So instead, you decide to add sugar to the container with your mixture where you can't stir thoroughly without fear. But that presents a math problem. You start by pouring a tablespoon of the sugarless chocolate mixture into a mold cavity to make the one sugar-free piece. But now you have to add some amount of sugar to the mixture so that when you fill the next cavity, it contains the correct amount of sugar, and so on.

3. Knowing that in the end, you want one piece with $k/2$ grams of sugar for each integer k from 0 to 10, inclusive, in what order (in terms of sugar content) should you make the chocolates? After filling each cavity, exactly how much sugar should you add into your mixture to make the next piece?

Let us know how many grams of sugar you need per tablespoon to make your perfect chocolate!

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 36 - Meet 5 March 6, 2025	Mentors:	Jade Buckwalter, Minerva Johar, Chloe Kim, Shauna Kwag, Jessie Lee, Hanna Mularczyk, Swathi Senthil, Dora Woodruff

Don't be afraid to make up words and symbols! When you do math, you're bound to have ideas for which you do not know a word or a notation. When that happens, just create a word or symbol for the idea. If you don't create a word or symbol for your idea, you will not easily be able to record your thoughts about the idea. If you take a break from the problem, unless you have an infallible memory (who does?), you'll end up having to refigure out those lost ideas.

When you decide on a word or symbol for the idea, first write down the meaning of your word or symbol. When you're exploring math, it can be very easy to edit one's ideas, and if you don't write down meanings, you might find yourself using the word in different ways each time it appears.

If you don't like the word or notation you came up with, you can always change it! And if it turns out that there already is a word for it, no worries! You'll eventually find out.

Session 36 - Meet 6 March 13, 2025	Mentors:	Elizabeth Bullock, Jade Buckwalter, Minerva Johar, Shauna Kwag, Jessie Lee, Maya Robinson, Swathi Senthil, Dora Woodruff

Let n be a positive integer. Can you find a connection between the following two sets? Let S be the set of 2 by n arrays containing the numbers 1 through $2n$, inclusive, each exactly once and in such a way that down any row or column, the numbers are in increasing order. Let T be the set of partitions of a regular $(n + 2)$-gon into n triangles using $n - 1$ line segments that join vertices of the polygon.

When $n = 3$, there are five elements in S:

1	2	3
4	5	6

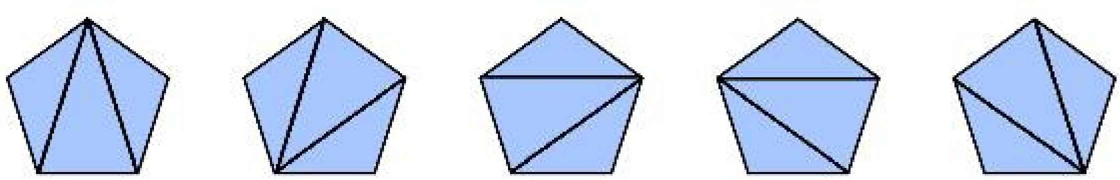
1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

And there are five elements in T:



If you're thinking that S and T have the same size, you'd be correct! Can you prove it?

Session 36 - Meet 7 March 20, 2025
Mentors: Elizabeth Bullock, Jade Buckwalter, Shauna Kwag, Yaqi Li, Swathi Senthil, Ella Wilson, Dora Woodruff, Saba Zerefa

Today, a member made progress on understanding the Navier-Stokes equation, which is a partial differential equation that models fluid flow. To understand Navier-Stokes, it is important to understand concepts from multivariable calculus, such as the gradient and the divergence.

Session 36 - Meet 8 April 3, 2025
Mentors: Elisabeth Bullock, Jade Buckwalter, Minerva Johar, Chloe Kim, Jessie Lee, Yaqi Li, Hanna Mularczyk, Maya Robinson, Swathi Senthil, Dora Woodruff

What is your favorite logo? Can you model it mathematically? For example, say you like the look of the capital "G" in "Google". Can you make a function $f(x, y)$ such that the border of the "G" looks like the plot of the points (x, y) such that $f(x, y) = 0$ and the points (x, y) such that $f(x, y) \leq 0$ constitute the filled in "G"? Suppose you are able to get 6 different functions $L_k(x, y)$, $k = 1, 2, 3, 4, 5, 6$, where the graph of $L_k(x, y) \leq 0$ is the kth letter in "Google". What single function $L(x, y)$ would produce the entire logo by graphing $L(x, y) \leq 0$?

Session 36 - Meet 9 April 10, 2025
Mentors: Elisabeth Bullock, Anika Bokil, Minerva Johar, Chloe Kim, Shauna Kwag, Jessie Lee, Hanna Mularczyk, Maya Robinson, Swathi Senthil, Dora Woodruff

Given a partial differential equation that models some physical phenomenon, how do you use it to create a simulation of what it models? For example, the differential equation

$$\frac{d^2}{dt^2} f(t) = -kf(t),$$

where k is a positive constant, models the motion of a mass hanging from a spring (with no dampening). Can you now write a computer program, for example, that simulates this motion in accordance with the equation?

Session 36 - Meet 10 April 17, 2025
Mentors: Elisabeth Bullock, Minerva Johar, Shauna Kwag, Yaqi Li, Hanna Mularczyk, Maya Robinson, Dora Woodruff

Suppose you have a sequence a_n where $a_n = p(n)$, where $p(x)$ is a polynomial of degree d in x, with $d > 0$. Can you prove that the sequence $b_n = a_{n+1} - a_n$ is a sequence whose terms are given by $q(n)$ where $q(x)$ is a polynomial of degree $d - 1$ in x? Suppose c_n is a sequence such that

$$a_n = c_{n+1} - c_n.$$

Can you show that c_n must be given by $r(n)$, where $r(x)$ is a polynomial of degree $d + 1$ in x? What are all the polynomials $r(x)$ such that the sequence $c_n = r(n)$ satisfies $a_n = c_{n+1} - c_n$?

Calendar

Session 35: (all dates in 2024)

September	12	Start of the thirty-fifth session!
	19	
	26	
October	3	
	10	
	17	
	24	
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	

Session 36: (all dates in 2025)

January	30	Start of the thirty-sixth session!
February	6	Cancelled - Weather
	13	
	20	No meet
	27	
March	6	
	13	
	20	
	27	No meet
April	3	
	10	
	17	
	24	No meet
May	1	
	8	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________
(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________