Girls' Angle Bulletin

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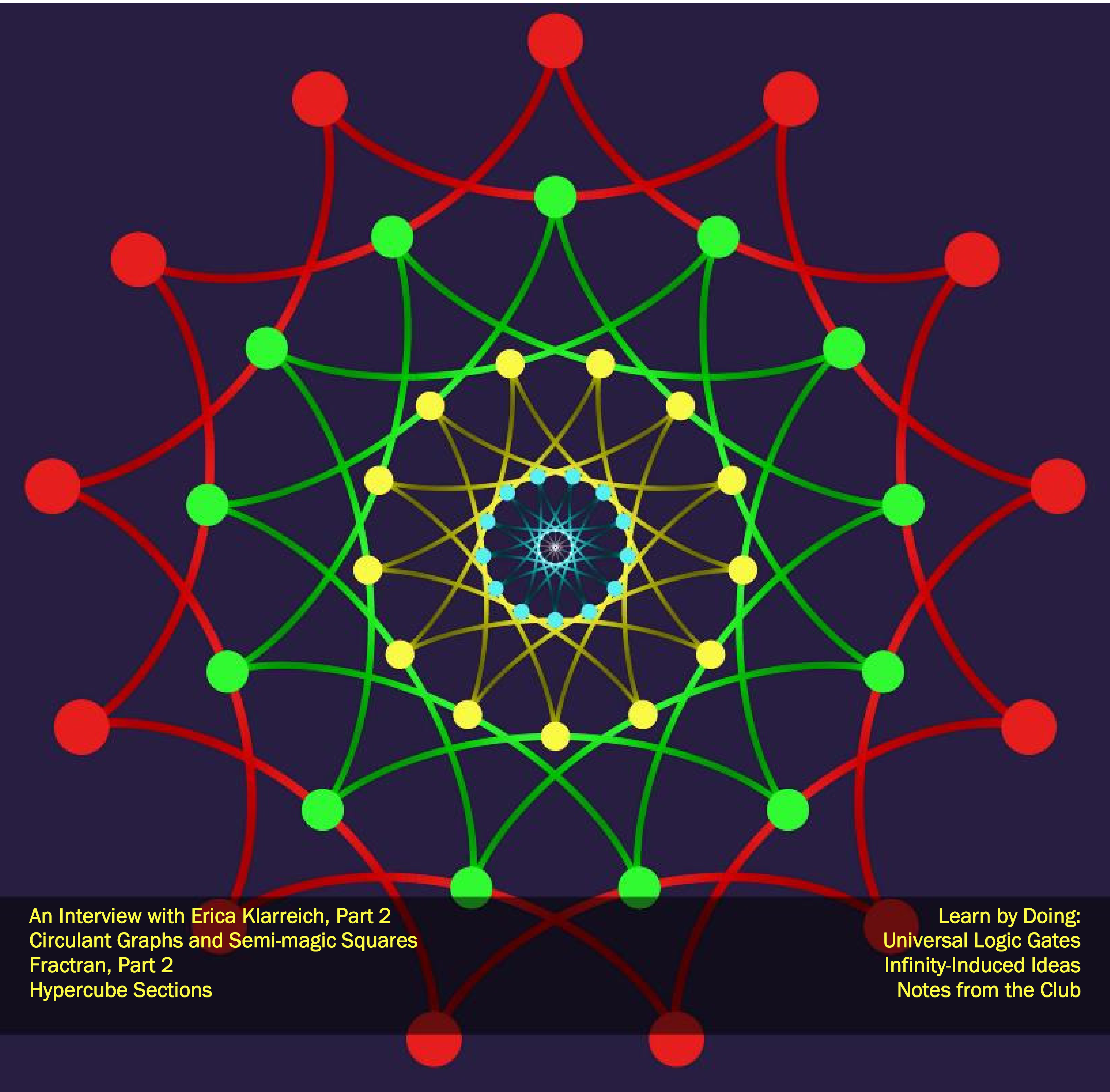
To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

When learning a new concept in math, think about it until it's plain and simple. Understanding the context in which it arises or how it's used helps. Try to "break" or modify the concept to see how its parts work. Don't worry, no ideas will actually get hurt. -Ken Fan, President and Founder

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Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Circulants* by Robert Donley and C. Kenneth Fan. See "Circulant Graphs and Semi-magic Squares" on page 7.

An Interview with Erica Klarreich, Part 2

I think it can be really helpful for people who aren't sure what kind of career they want to pay attention to what they enjoy…

We pick up Part 2 in the middle of Erica's remarks about turning mathematics into words.

Erica: Mathematics papers are almost uniquely hard to read if you're outside of the particular field that they're written in. If I wanted to sit down and read a paper in, say, genetics or astronomy, I probably wouldn't get that much out of the paper, but I would get something out of it. I might be able to write a few sentences about it.

But I think most people, if they look at a math paper, would not even be able to write a few sentences explaining what it's about. That's partly because science papers have a very specific format: a science paper has an introduction, then it will talk about the methods and results, then it will end with a discussion. You can almost always get something out of the introduction and the discussion.

Mathematics papers don't have that convention. There's often no discussion at the end. The end of the paper might be proof of corollary 3.7; you get to the end of the proof, and that's the end of the paper. And the introductions can be hard to read as well.

Mathematics is basically a process of abstraction. You may understand some examples of a phenomenon, and then you abstract it so you can understand it in broader settings. In doing so, you sometimes find tools that are very powerful and effective, but that you could only see once you abstracted it and made it look cleaner.

I think because of that, mathematicians tend to feel as if the writing of mathematics should be similarly abstract. If they find an abstraction that is very powerful, then they'll write the paper and say, "Here is this abstract thing and here is what I could prove about it." What you don't see is any of the thoughts that led to that abstraction, like the simple examples that made you want to think about this abstract thing and that informed how you thought about it. You don't show any of that, you just show this very clean thing that you write about in the most concise way.

There's this style in mathematical writing where you try to formulate everything in the most general way and write it in the tersest way possible. And I don't think that's a very effective way to communicate. I think it would be nice if mathematicians wrote longer papers, where maybe the result would be expressed in that extremely terse and super general formulation, but there would also be sentences that are written in some approximation of real English like, "So, when you see this, you can kind of imagine that this thing here is the integers," or "Here's a baby example of this thing that we're creating. If you keep this example in mind, you should get all the most salient features, although there a few technical ways in which what we're doing is different."

Ken: As you speak about this, I'm thinking about your writings and how quickly you engage the reader – me, in this case – with the story; you say things that I can relate to and get me thinking about these things. Then I feel personally invested in the story.

Erica: It feels like there is almost some kind of cultural taboo about writing in the way that I just described; so if you did write it in that way, people would think that you were not a serious mathematician. But when I'm writing, I make liberal use of words like "roughly," "essentially," or "more or less." This lets readers know that the actual thing is a little more complicated, but that this is a good example to keep in mind. I, as a journalist, feel free to use those words whenever I want to. But I think mathematicians feel very uncomfortable with that sort of thing. They don't ever want to say something that is not 100 percent correct, even if it could help people develop some intuition for the thing that *is* 100 percent correct.

I remember someone talking to me about how mathematics is really an oral tradition. This whole business of writing papers is actually more about prestige and benchmarks. You have to write the paper so that it's absolutely clear that you proved what you claim to have proved, and anyone who wants to check it can check it. And then you get this publication, and you can put it on your CV and it will help you in your career. But the way that mathematics is actually transmitted is orally.

Just the other day, I was talking to a mathematician about some work that's being done in a field that's adjacent to hers. She said now that they've finally written up their big results, they'll be more available to talk to mathematicians who are a little bit outside of what they do. I think, for a lot of mathematicians, that's the way that real understanding happens: either through these one-on-one conversations, which are often the most effective way to understand something, or through seminar talks where there can be a real interaction, a back and forth. I think that many mathematicians, if they want to learn about some area, will turn to papers as a last resort, if there's just no one available with whom they can speak about it.

Ken: I totally relate to everything you just said. I also think that it makes your writing all the more remarkable; you're probably right that it is more of an oral tradition. Yet you write about math so well! People can get a lot from reading your papers, and maybe like you said, the audience is different. But, I wonder if professional mathematicians might actually derive some benefit by reading your papers on topics that they are not expert in. Perhaps they could gain a context for another mathematician's technical paper by reading your account of the work which would help them stay oriented as they read the technical paper.

Actually, what you were saying about using words like "roughly" and other indications that you're not being completely precise leads me to another question I wanted to ask. I know that a lot of people who try to describe scientific or mathematical topics use analogies that are not exact analogies. There's always something about an analogy that is misleading, like using a ball rolling on a stretchy surface and watching it sag to describe the theory of general relativity, or talking about the fourth dimension or a four-dimensional hypercube and showing a tesseract, which is really a projection of it. How do you navigate this territory? What kind of compromises on the exact truth are you willing to make for the purposes of communication and to ensure that you don't end up misleading others? How do you handle that?

Erica: You're asking very challenging questions, which is good.

As you know, I'm originally a mathematician myself, so I – like other mathematicians – feel very uncomfortable if I write something that is kind of wrong. I'll take one of your examples. You were talking about a ball weighing down space time, and

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Circulant Graphs and Semi-magic Squares[1]

by Robert Donley[2]
edited by Amanda Galtman

In this installment, we bring together many of the ideas developed in the last five installments. In particular, we apply the path model for the construction of Young diagrams from Volume 17, Number 3 to semi-magic squares to obtain regular edge colorings. Recall that we constructed Young diagrams by adding one square at a time; this resulted in a sequence of Young diagrams, which in turn was recorded as a standard Young tableau. In this installment, we construct a regular edge coloring on a fixed even number of vertices, by adding one perfect matching at a time. We also record the sequence in a Latin rectangle.

First, let's warm up with a new type of graph.

Definition: A circulant matrix A is a square matrix that is constant along each diagonal, if we allow diagonals to wrap across the borders. In other words, with a given first column, we define successive columns by shifting each column entry to the right by one and down by one, wrapping from the last row to the first row.

If we move the first k columns of the identity matrix to the right-hand side, we obtain a circulant matrix. Any linear combination of such matrices is also circulant. For example, consider the following circulant matrices of size three:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} = I + 2P_1 + 3P_2 .$$

In general, a circulant matrix of size 3 has the form $\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$.

Any cyclic shift of columns of the identity matrix is a special case of a permutation matrix, as any permutation matrix is obtained by permuting the columns of the identity matrix.

Exercise: Give an example of a permutation matrix that is not a circulant matrix.

Exercise: Find all circulant matrices of sizes 3, 4, 5, and 6 with entries equal to 0 or 1. Give a formula for the number of such circulant matrices of size n.

Definition: A circulant graph G is a graph whose adjacency matrix is a circulant matrix.

In general, the definition of a circulant graph does not include the connectedness condition. Since our graphs do not have loops or multiple edges, the only nontrivial circulant graph of size three

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[1] This installment is 19th in a series that began in Volume 15, Number 3. It is also part 6 of a subseries that began in Volume 17, Number 4.

[2]This content is supported in part by a grant from MathWorks.

is the 3-cycle, which has adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Exercise: From the set of all circulant matrices of sizes 4, 5, and 6 that are also adjacency matrices, draw the corresponding graphs. Which of these adjacency matrices correspond to graphs that are equivalent to each other?

In general, suppose C_k is the cyclic shift of the identity matrix of size n by moving the first k columns to the right-hand side. In addition to the identity matrix $C_0 = I$, C_m is a symmetric circulant matrix when $n = 2m$. For other k, $C_k + C_k^T$ is a symmetric circulant matrix with entries 0 or 1. Note that $C_k^T = C_{n-k}$, and, in general, the graph corresponding to $C_k + C_k^T$ is a disjoint union of cycles.

Exercise: When $n = 6$, find k such that the graph corresponding to $C_k + C_k^T$ is not connected. Formulate a sufficient condition for a circulant matrix to correspond to a connected graph.

Exercise: When $n = 2m$, draw the graphs corresponding to C_m and $C_1 + C_1^T$. Describe these graphs in plain language.

Exercise: Suppose $n > 2$ is a prime. Describe the graph of $C_k + C_k^T$ for each $1 \le k < n - 1$.

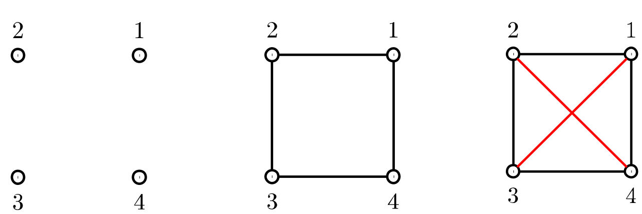
Next, we construct circulant graphs using a path model. If we start with n vertices, then the adjacency matrices have size n. The graph with no edges has the zero matrix for its adjacency matrix. We include edges by adding terms of the form $C_k + C_k^T$ and, when $n = 2m$, C_m. In turn, we record the sequence of edge additions as an adjacency list, read from top to bottom. The adjacency list has the Latin rectangle property. This process terminates at a complete graph and Latin square.

Example: Consider the sequence of adjacency matrices $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$,

with corresponding adjacency lists as Latin rectangles

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline \end{array} \quad \to \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{array} \quad \to \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \end{array}$$

and corresponding graphs

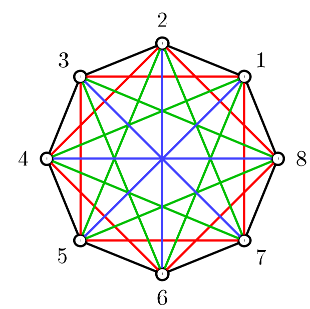


For the Latin rectangle property, the top row corresponds to the vertex labels, and the lower rows correspond to C_1^T, C_1, and C_2 in descending order. Note that these Latin rectangles do not correspond to regular edge colorings, although these graphs admit regular edge colorings with different Latin rectangles.

In general, a circulant graph with term $C_1 + C_1^T$ contains an n-cycle which can be depicted as the boundary of a regular n-sided polygon if we label the vertices 1 through n in clockwise or counterclockwise order. To implement the circulant property for additional cycles, we choose a gap number g, and connect each vertex to the vertex at g places counterclockwise. We can repeat this procedure with several g.

Exercise: Prove that the edge set corresponding to g forms a disjoint union of cycles. Here we include an isolated edge as a 2-cycle. Describe the cycle structure of this edge set based on the greatest common divisor of n and g. In particular, note when n and g have no common factors and when g divides n.

Exercise: With the above method, draw all circulant graphs on 4, 5, and 6 vertices, and compare with the earlier exercise on circulant matrices.



Exercise: With this method, draw all circulant graphs on a regular heptagon. Find the adjacency matrix for each graph.

Exercise: Determine the gap used for each color on the octagon at the right. Then construct the corresponding colorings for the nonagon, the decagon, and the dodecagon.

For an alternative model based on modular arithmetic, recall that, for $n \geq 2$, the integers modulo n are the set $\mathbb{Z}/n\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$, where for each integer k, the element $\overline{k}$ in $\mathbb{Z}/n\mathbb{Z}$ is the set of integers that leave the same remainder as k upon division by n. In $\mathbb{Z}/n\mathbb{Z}$, we have $\overline{a} + \overline{b} = \overline{a+b}$. Then, for a given gap g, a cycle corresponds to the repeated addition of $\overline{g}$ to a fixed element. When we work modulo n, we will omit the bar over the numbers.

Example: For the octagon, repeatedly adding 3 to 1 corresponds to the green cycle

$$1 \to 4 \to 7 \to 2 \to 5 \to 8 \to 3 \to 6 \to 1.$$

Note that, in modular arithmetic, $0 = 8 \pmod 8$.

Exercise: Redo the previous three exercises using modular arithmetic.

To interpolate between the circulant graphs and modular numbers, we describe circulant graphs in terms of complex numbers. Recall Euler's formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta),$$

and consider the points with $\theta = 2\pi k/n$ for $1 \leq k \leq n$ as the vertices of a regular n-sided polygon in the complex plane. For a fixed gap g, we draw an edge when multiplication by $e^{2\pi g i/n}$ carries one vertex to another.

Exercise: How does this model incorporate modular arithmetic?

Exercise: If a set is available near you, explore the Spirograph design set. Many of the patterns correspond to a circulant graph if we place a vertex at the outermost point of each leaf.

Exercise: Explore what happens if we use Toeplitz matrices instead of circulant matrices. That is, suppose the values on diagonals are constant but do not need to continue past the border.

Exercise: Draw the Hasse diagrams for circulant matrices of size 4, 5, and 6 if the partial ordering is defined by entry-wise comparison of matrices. What is the difference between the odd and even cases?

For the remainder of this installment, our goal is to develop a general path model for regular edge colorings. We first develop some basic properties of semi-magic squares.

Definition: A square matrix M of size n is called a **semi-magic square** with line sum L if the entries are nonnegative integers and the sums along each row and column are equal to L.

Example: The zero matrix is a semi-magic square with line sum $L = 0$, and every permutation matrix is a semi-magic square with line sum $L = 1$.

Exercise: Prove that the sum of two semi-magic squares of the same size is a semi-magic square. Prove that a multiple of a semi-magic square by a nonnegative integer k is a semi-magic square. What are the line sums in each case?

Exercise: Prove that the semi-magic property and line sum are preserved by row permutations, column permutations, and matrix transpose.

Exercise: Prove that the adjacency matrix of a circulant graph is a semi-magic square. What is the line sum?

From an exercise above, we see that every sum of permutation matrices is a semi-magic square with line sum equal to the number of permutation matrices in the sum. The converse of this statement is true, as a corollary to the Birkhoff-von Neumann theorem:

Corollary to the Birkhoff-von Neumann theorem: Every semi-magic square with line sum L is the sum of L permutation matrices.

This sum need not be unique, which makes semi-magic squares both fun and difficult to use.

Exercise: Write the following semi-magic squares in two different ways as sums of permutation matrices. Each sum should use distinct sets of permutation matrices.

$$J_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Exercise: As an alternative to guesswork for J_3 in the previous exercise, solve the equation

$$a_1P_1 + \dots + a_6P_6 = J_3,$$

where the sum ranges over the six permutation matrices P_i of size 3. Describe the outcome in terms of number of column switches of the identity matrix.

Exercise: Prove the corollary for semi-magic squares of size 3.

If an adjacency matrix is a semi-magic square, then it is a sum of permutation matrices, but in such a way that no entries equal to 1 occur in a common position. Thus, these permutation matrices can be encoded as the rows of a Latin rectangle. On the other hand, there is no guarantee that these permutation matrices can be chosen to be involutions, a requirement for the existence of a regular edge coloring.

Exercise: What basic graph property is guaranteed when an adjacency matrix is a semi-magic square? What does the line sum measure?

To simplify our approach, we assume that our graphs are bipartite on an equal number of vertices. When we order the vertices by listing one subset of the bipartition first, the adjacency matrices for such graphs have the form

$$A_M = \begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}.$$

If $n = 2m$, then 0 and M represent square matrices of size m.

Exercise: Prove that A_P corresponds to an involution if and only if P is a permutation matrix.

Thus, if A_M is a semi-magic square with line sum L, then M is a sum of L permutation matrices and A_M is a sum of L involutions.

Exercise: When does a circulant graph admit a regular edge coloring?

We now have a path model to construct regular edge colorings on bipartite graphs. In fact, the corollary gives a model with many of the interesting features of the Young lattices $L(m, n)$.

First, the set of all such A_M matrices admits a partial ordering by entry-wise comparison. When $M = 0$, the corresponding graph is just a set of n vertices. When $M = J$, where J is the matrix with every entry equal to 1, the graph is the complete bipartite graph $K_{m,m}$. These are the minimum and maximum entries for the partial ordering.

Next, define the complement $A_{M'}$ of A_M to be the adjacency matrix where the entries of M are changed from 0 to 1, and vice versa.

Exercise: Prove that M' is a semi-magic square. If the line sum of M is L, what is the line sum of M'? Describe the graph corresponding to $A_{M'}$.

Covering relations are described as follows: A_M is covered by A_N ($A_M \lessdot A_N$) if and only if $M = N + P$ for some permutation matrix P. To construct a saturated chain from A_0 to A_M, we simply add permutation matrices to the zero matrix until we obtain M. If we number the nonzero entries of the permutation matrix with a k at the kth step, we obtain an adjacency matrix for a regular edge coloring of the graph for A_M. If we append each permutation matrix in one-line notation to the adjacency list, we obtain a Latin rectangle.

On the other hand, every saturated chain starting at the minimum and ending at A_M can be completed to a maximal chain ending at A_J, the adjacency matrix of $K_{m,m}$. Since M' is a semi-magic square with entries of only 0 and 1, or a **0/1-semi-magic square**, it equals a sum of permutation matrices with no common 1s. Thus, we obtain a Latin rectangle for M'.

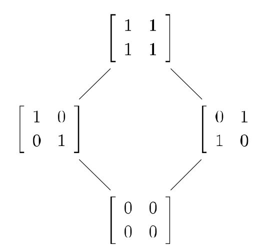
Exercise: Prove that appending a Latin rectangle for M' to the bottom of a Latin rectangle for M results in a Latin square.

Exercise: Complete the Latin rectangle at the right to a Latin square. First, complete it by trial and error. Then record the corresponding M and write M' as a sum of permutation matrices. How many completions can you find?

$$\begin{array}{cccc} 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{array}$$

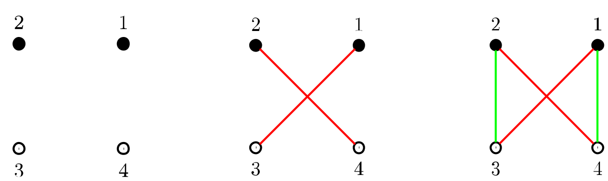
Overall, we obtain a finite graded partial ordering with a unique minimum and maximum, just like we saw with the Young lattices $L(m, n)$.

Example: When $m = 2$, we have the Hasse diagram for the adjacency matrices A_M, each represented by M, at the right. The maximal chain on the left-hand side of the diagram is represented by Latin rectangles as follows:



$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline \end{array} \rightarrow \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 1 & 2 \end{array} \rightarrow \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

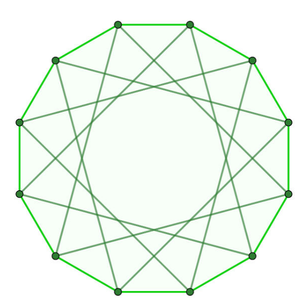
As with standard Young tableaux, the final Latin rectangle describes the entire path. Finally, the sequence of regular edge colorings, which ends in $K_{2,2}$, is given by



Exercise: Repeat the example for the maximal chain on the right-hand side of the Hasse diagram.

Exercise: Repeat the previous example for $m = 3$. How many distinct regular edge colorings are there, up to vertex labeling and edge colors?

For $m = 4$, there are 140 0/1-semi-magic squares. In general, the number of 0/1-semi-magic squares grows quickly with L; this sequence is entry A067209 at the On-Line Encyclopedia of Integer Sequences (oeis.org) and begins 1, 2, 4, 14, 140, 4322, 434542,…. In the next installment, we will use group theory and the orbit-stabilizer theorem to count both the 0/1-semi-magic squares and the Latin squares for $m = 4$, 5, and 6.



A 12-node circulant graph.

Fractran, Part 2

by Ken Fan | edited by Jennifer Sidney

Emily: Working out a fractran program to test for divisibility definitely helped me get a feel for how fractran works. I'm reasonably confident that we could write a prime number-generating fractran program. I guess we could look at Conway's program and try to figure out how it works.

Jasmine: Actually, we probably have some idea of how it works already. It seems to use the exponent of 2 as a counter, which I'll just call N. It tests this counter for primality, probably by checking if the number is divisible by a number strictly between 1 and N; if it isn't, it clears all the other counters and flags so that it outputs this power of 2 whose exponent is a prime number. Whether it finds a divisor or not, it then increases N by 1 and repeats.

Emily: That sounds plausible. At least, that's what I would try to do if we were to write a prime-generating fractran program. Do you feel confident that we'd be able to write such a program?

Jasmine thinks for a moment.

Jasmine: Fairly confident, but one thing that would make me a lot more confident is if we could see a way to break an algorithm up into pieces and then stitch these pieces back together in fractran. In other computer languages, there are usually nice ways to split the code up into small, manageable chunks, then gather these code snippets together into a larger program. If we could do that in fractran, then I'd feel confident that we could implement any algorithm in that language.

Emily: You mean like the way some programming languages let the programmer define functions whose code can focus on performing one task without worrying about what the rest of the program does? And when the function is invoked, its code is executed without interfering with whatever the rest of the program is doing?

Jasmine: Yes, exactly. Can that be done in fractran?

Emily: If that can be done, it would make programming in fractran much easier! I have a feeling that can be accomplished by using lots of flags. The good news is that there are infinitely many prime numbers, so we will never run out of unused flags or counters. We will never have to worry about having to reuse a prime number to hold information for two or more different purposes.

Jasmine: I suppose if we have a fractran code snippet that we want to incorporate in a larger program, we can pick some unused prime number, let's call it F_1, and add that factor of F_1 to the denominator of every fraction in the code snippet. Turning the flag F_1 on activates this code snippet. We'd do a similar thing for every code snippet. Each snippet would be associated with a unique flag, so the fractions in it could only execute if its unique flag is activated.

Emily: Of course, every time a fraction in the code snippet executes, the flag will be turned off. So the code snippet needs to reactivate F_1 as long as it wants to keep on running. So if one of its fractions executes and the program should stay within this code snippet, we can add another flag (corresponding to some different, unused prime number and also unique to this code snippet) F_2 to the numerator. Then, near the beginning of the code, we can place the fraction F_1/F_2 so that the first thing that will happen if F_2 is set is to turn off F_2 and turn F_1 back on.

Jasmine: I think this scheme would work. So if we have a bunch of code snippets, we can assign each a unique pair of flags, an identifier F_1 and a reactivator F_2. If the identifier flag F_1 is on, that tells the fractran program to execute the corresponding code snippet. No other code snippet will activate because among the collection of identifier flags, we just make sure that only one is active at a time. The active code snippet will keep turning on the reactivator flag F_2 as long as it wants to keep executing. When it is done with its computation, instead of turning on the reactivator flag F_2, it will turn on the identifier flag for whatever snippet of code it would like the program to switch over to.

Emily: Or, it could activate a flag associated with a prime number that is not in any denominator, in which case the program will terminate.

Jasmine: Yes. I feel quite confident now that any program could be implemented in fractran.

Emily: Actually, to really settle this matter, we could verify that we can implement one of the universal logic gates[1] in fractran, such as "not and."

Jasmine: Of course! Great idea! If we can implement "not and," then we could use the flag method to program in fractran any algorithm that a computer can execute. Let's make a "not and" gate in fractran!

Emily: Okay! Let's say that B_1 and B_2 are the inputs, and the output is O. Let's also include our flagging scheme, so let's let F_1 be the active flag and let F_2 be the flag that says, "reactivate the active flag F_1."

Jasmine: And let's say that X is the flag corresponding to the code that should execute when the "not and" gate code is finished.

Emily: When the code is done, I think we should make sure that B_1 and B_2 are restored to whatever they were at the start, O should be the result of applying "not and" to B_1 and B_2, both F_1 and F_2 should be 0, and X should be set to 1.

Jasmine: Agreed.

Emily: So O should end up being 0 if both B_1 and B_2 are 1; otherwise, it should be 1.

[1] For more on universal logic gates, see this issue's "Learn by Doing" on page 23.

Jasmine: How about this? We first make sure that O is 0. Then, if B_1 and B_2 are both 1, we flag this condition, restore B_1 and B_2, and exit. Otherwise, we increase O, then exit.

Emily: That sounds good. First, we need our F_1 reactivation fraction F_1/F_2. Then we can put $F_2/(OF_1)$ so that the program first reduces O to 0.

Jasmine: If it gets past that fraction, we know O is 0, so we can put $F_3F_2/(B_1B_2F_1)$ next. If this executes, it means both B_1 and B_2 are 1, so we don't have to increment O. We just need to restore B_1 and B_2 to 1 and exit. That can be accomplished with $B_1B_2X/(F_3F_1)$.

Emily: And if the code gets past *that* fraction, then we know that at least one of B_1 and B_2 must be 0, and possibly both, so we want to increase O and exit. We don't have to restore B_1 or B_2 because neither would have been changed. So we need the fraction OX/F_1. I think that does it!

Jasmine: Thus our entire "not and" code snippet is

$$F_1/F_2 \quad F_2/(OF_1) \quad F_3F_2/(B_1B_2F_1) \quad B_1B_2X/(F_3F_1) \quad OX/F_1$$

Emily: Looks good! Just to make sure it really works, let's trace it through for all four possible inputs. We can assume that F_1 is 1, but X, F_2, and F_3 are all 0. The value of B_1 could be 0 or 1, as could the value of B_2. And O can be anything.

Jasmine: If O is greater than 0, then the second fraction will execute, decreasing O and setting F_1 to 0, but setting the flag F_2. Then with the next step, F_2 will be reset to 0 while F_1 will be reactivated to 1. These two fractions will repeat until O is 0. So we can now proceed knowing that F_1 is 1 and X, F_2, F_3, *and* O are all 0. The values of B_1 and B_2 are unaffected by this process.

Emily and Jasmine create the tables on the next page that trace the values of the fractran counters as they step through their fractran program from the point where O is guaranteed to be 0 for the four different possible inputs.

Emily: They all work as we planned!

Jasmine: Yes, now I'm convinced that any algorithm can be implemented in fractran.

Emily: Do you feel like looking at Conway's prime generating program now?

Jasmine: Actually, not really.

Emily: That's funny, neither do I!

Jasmine: I feel like we've gotten a good sense of how fractran works, and we know that any computer algorithm can be implemented in fractran. So I'm no longer surprised that Conway's program exists.

Emily: Yes. Also, there are many different algorithms for generating prime numbers.

Jasmine: Well, what would you like to do, then?

Emily: Go to Cake Country!

Fractran code to compute B_1 NAND B_2

$\boldsymbol{F_1/F_2 \quad F_2/(OF_1) \quad F_3F_2/(B_1B_2F_1) \quad B_1B_2X/(F_3F_1) \quad OX/F_1}$

Executed Fraction	B_1	B_2	O	F_1	F_2	F_3	X
Starting Values	0	0	0	1	0	0	0
OX/F_1	0	0	1	0	0	0	1

Executed Fraction	B_1	B_2	O	F_1	F_2	F_3	X
Starting Values	0	1	0	1	0	0	0
OX/F_1	0	1	1	0	0	0	1

Executed Fraction	B_1	B_2	O	F_1	F_2	F_3	X
Starting Values	1	0	0	1	0	0	0
OX/F_1	1	0	1	0	0	0	1

Executed Fraction	B_1	B_2	O	F_1	F_2	F_3	X
Starting Values	1	1	0	1	0	0	0
$F_3F_2/(B_1B_2F_1)$	0	0	0	0	1	1	0
F_1/F_2	0	0	0	1	0	1	0
$B_1B_2X/(F_3F_1)$	1	1	0	0	0	0	1

Hypercube Sections

by Addie Summer | edited by Amanda Galtman

While trying to understand the symmetries of an n-dimensional hypercube, I stumbled upon a neat stratification of its vertices. If I choose any one of the vertices of the hypercube, then the vertices stratify themselves according to their distance from the chosen one. I think it's neat because this stratification organizes an exponential number of vertices (2^n) into just a linear number ($n + 1$) strata.

Like last time, let's take the hypercube to be the points in n-dimensional space with coordinates $(x_1, x_2, x_3, \ldots, x_n)$, where $-1 \le x_k \le 1$ for $k = 1, 2, 3, \ldots, n$. Then the vertices of the hypercube are the points each of whose coordinates is either 1 or -1.

If we take v to be the vertex whose coordinates are all equal to 1, then for each integer k from 0 to n, the vertices that have exactly k coordinates equal to -1 form one of the strata in the stratification. That's because the distance between v and such a vertex is $2\sqrt{k}$, as can be seen by applying the distance formula. Let V_k be these vertices.

The convex hull of the vertices in V_k is some kind of high-dimensional polyhedron, that is, a "hyperpolyhedron." I'm super curious to know what these shapes are!

Let's use the symbol C_k for the convex hull of the vertices in V_k. If there's ambiguity about what n is, I'll write $C_{k,n}$ and $V_{k,n}$.

When n is small, we can see what the shapes $C_{k,n}$ are.

When $n = 1$, there's only one coordinate, and the hypercube is a line segment. $V_0 = \{v\}$ and $V_1 = \{-v\}$, so both C_0 and C_1 are points.

When $n = 2$, our hypercube is a square. $V_0 = \{v\}$ (as it always is!), V_1 consists of the two vertices connected to v by an edge, and V_2 contains only the vertex $-v$, the vertex diagonally opposite v. Thus, C_0 and C_2 are points, but C_1 is a line segment stretching across a diagonal of the square.

When $n = 3$, we have a cube. Again, $V_0 = \{v\}$ and $V_3 = \{-v\}$, so C_0 and C_3 are points. But now V_1 consists of the three vertices connected to v: (1, 1, -1), (1, -1, 1), and (-1, 1, 1). The distance between any two of these points is $\sqrt{8}$, so C_1 is an equilateral triangle. The same is true of the three vertices in V_2, so C_2 is also an equilateral triangle.

At this stage, I'm contemplating a choice: I can continue on to the case $n = 4$, or I can try to make general observations about all the $C_{k,n}$. I guess there's no way to know beforehand which idea is better. Maybe both ideas are equally good. For variety's sake, I'll switch to thinking about the general situation.

The examples suggest that C_0 and C_n are always going to be points. And this is true. Only one point has distance 0 from v, namely, v itself. Also, only one vertex maximizes the number of its coordinates that are equal to -1, namely, $-v$. So, $V_0 = \{v\}$ and $V_n = \{-v\}$.

From now on, let's assume that $0 < k < n$.

The vertices in $V_{k,n}$ all have k coordinates equal to -1 and $n - k$ coordinates equal to 1. That means that the sum of the coordinates is equal to $n - 2k$. In other words, if $(x_1, x_2, x_3, \ldots, x_n)$ is in $V_{k,n}$, then $x_1 + x_2 + \ldots + x_n = n - 2k$, which is the equation of an $(n - 1)$-dimensional hyperplane. This means that V_k is not only contained in the hypersphere centered at v (with radius $2\sqrt{k}$), but also contained in a hyperplane. That tells us that $C_{k,n}$ is at most $(n - 1)$-dimensional, since all the points that define it sit inside an $(n - 1)$-dimensional hyperplane.

It's interesting that all the vertices in $V_{k,n}$ sit in both a hypersphere and a hyperplane, because the intersection of our $(n - 1)$-dimensional hypersphere and this $(n - 1)$-dimensional hyperplane is an $(n - 2)$-dimensional hypersphere.[1] Since a hyperball is a convex shape whose boundary contains no planar sections, every vertex in $V_{k,n}$ must be a vertex of its convex hull $C_{k,n}$. Thus, $C_{k,n}$ is a hyperpolyhedron with ${}_nC_k$ (i.e., "n choose k" or $n!/(k!(n-k)!)$) vertices. We found a family of hyperpolyhedra whose vertices are counted by Pascal's triangle. Neat!

What can we say about the dimension of $C_{k,n}$? We know $C_{k,n}$ lives inside an $(n - 1)$-dimensional hyperplane, but could its dimension be smaller than $n - 1$? If it were, it would mean that all the vertices in V_k also lie inside a hyperplane different from $x_1 + x_2 + \ldots + x_n = n - 2k$. Let's see if we can find one! Suppose that

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = d$$

is the equation of a hyperplane satisfied by the coordinates of every vertex in V_k. The way the coordinates of the points in V_k differ is in which subset of k coordinates is equal to -1. Changing that subset could potentially change the sum on the left-hand side of the equation of our hyperplane, but we need that sum to *not* change. It must be equal to d for every point in V_k. This suggests looking at what conditions we must impose on the coefficients to meet the demand that that sum, $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n$, not change if we swap a coordinate equal to 1 with a coordinate equal to -1.

So let $p = (x_1, x_2, x_3, \ldots, x_n)$ be a vertex in V_k, let i be an index where $x_i = 1$, and let j be an index where $x_j = -1$. (We can always find such coordinates since $0 < k < n$, i.e., not all the coordinates equal 1 and not all the coordinates are equal to -1.) The point whose coordinates are the same as those of p except that we change the sign of the ith and jth coordinates is also a vertex in V_k because it still has k coordinates equal to -1. But changing the sign of the ith and jth coordinates would change the value of $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n$ by $2a_j - 2a_i$. Since this sum shouldn't change at all, we must have $2a_j - 2a_i = 0$, which means that $a_j = a_i$. And since there is a vertex in V_k whose ith coordinate is +1 and whose jth coordinate is -1 for any i and j that aren't the same, it must be that $a_1 = a_2 = a_3 = \ldots = a_n$. Since the coefficients cannot all be equal to 0, we can divide by this common value of the coefficients to see that any hyperplane that contains all the vertices in V_k can be written in the form

$$x_1 + x_2 + \ldots + x_n = d/a_1.$$

[1] In general, a hyperplane can be tangent to the hypersphere or misses the hypersphere entirely. In those cases, the intersection is a single point or empty, respectively. In our situation, we know that for $0 < k < n$, there is more than one point in the intersection, so the hyperplane is not tangent to the hypersphere. When $k = 0$, there's only one point in the intersection, but in this case, the hypersphere has radius 0 and degenerates to a point. When $k = n$, there's again only one point, and the hypersphere is, indeed, tangent to the hyperplane.

By substituting the coordinates of any vertex in V_k into the equation, we find that d/a_1 must be equal to $n - 2k$. The hyperplane we're examining is the very hyperplane we are trying to avoid. That means that there is no other hyperplane that contains all the vertices in V_k, so the hyperpolyhedron $C_{k,n}$ *must* be $(n - 1)$-dimensional (for $0 < k < n$)!

Progress!

When $n = 4$, the shapes C_1, C_2, and C_3 must be 3D polyhedra. I'm so curious to know what solids they are. I'll go back to working out examples to figure that out.

We know that C_1, C_2, and C_3 have 4, 6, and 4 vertices, respectively.

In three dimensions, a polyhedron with 4 vertices has to be a triangular pyramid (a.k.a. tetrahedron). We saw that we can permute the vertices in V_1 however we wish by some symmetry of the hypercube, so, in fact, C_1 must be a regular tetrahedron. (This can also be deduced by applying the distance formula to any pair of vertices in V_1, but isn't it nice to use previous observations to spare us the computation?)

Now, C_3 is the image of C_1 under the transformation that sends x to $-x$, so C_3 is also a regular tetrahedron.

That leaves $C_{2,4}$. I suppose we could try to make a very careful sketch of the object by placing a 3D-coordinate system within the 3D-hyperplane $x_1 + x_2 + x_3 + x_4 = 0$, then working out the coordinates of each vertex in this coordinate system, and then using those coordinates to make a drawing or even a model. But when we explore the higher-dimensional cases, we won't have this option. So let's try to figure out what $C_{2,4}$ is without a drawing.

Since we know that there are enough symmetries of the hypercube to send any specific vertex of $C_{2,4}$ to any other vertex, all the vertices of $C_{2,4}$ must look alike. That is, they must have the same number of edges emanating from them and they must be the corners of the same configuration of faces. (We cannot, however, say that the distance between vertices is the same for any pair of vertices, as we did in the case of C_1. While we can map any vertex to any other vertex using a symmetry of the hypercube, we cannot map any *pair* of vertices to any other pair of vertices.)

What are the distances between pairs of vertices in $V_{2,4}$? Using the distance formula, we can see that the distance between any two vertices in $V_{2,4}$ is either $\sqrt{8}$ or 4, depending on whether they share a coordinate that is equal to -1. For each vertex, there is only one other vertex where the coordinates that are equal to -1 are completely disjoint. Therefore, each vertex is $\sqrt{8}$ units away from four vertices and 4 units away from one vertex.

We can organize the vertices of $C_{2,4}$ into three pairs where, in each pair, the distance between the vertices is 4. If P is a vertex in $V_{2,4}$, then $-P$ is the vertex 4 units away. So, we can say that $V_{2,4} = \{P, -P, Q, -Q, R, -R\}$. The distance between any two of these vertices labeled by different letters is $\sqrt{8}$. That means that the triangle whose vertices are P, $-P$, and X, where X can be Q, $-Q$, R, or $-R$, has side lengths $\sqrt{8}$, $\sqrt{8}$, and 4. These are the side lengths of an isosceles right triangle! So, the two triangles with vertices P, $-P$, Q and P, $-P$, $-Q$ are congruent isosceles right triangles. Furthermore, since the distance between Q and $-Q$ is 4, these two triangles must lie in the same plane; they do not hinge along their common edge with endpoints P and $-P$. Thus, the 3 sets $\{P, -P, Q, -Q\}$, $\{P, -P, R, -R\}$, and $\{Q, -Q, R, -R\}$ represent three squares, any two of which intersect along a diagonal.

That shape is a regular octahedron!

Working out the shapes associated with $n = 4$ gave rise to a number of facts. Let's see if those facts hold in general.

For example, for any n, the transformation that sends a point x to $-x$ is an isometry that sends V_k to V_{n-k} and so sends C_k to C_{n-k}. Thus, by symmetry, C_k is congruent to C_{n-k}.

Also, $C_{1,n}$ and $C_{n-1,n}$ must both be the generalization of the regular tetrahedron to higher dimensions, which is known as a **regular simplex**. Both $C_{1,n}$ and $C_{n-1,n}$ have n vertices, which is the minimum number of vertices needed for a hyperpolyhedron to have dimension $n - 1$. And because we can permute these vertices however we wish using a symmetry of the hypercube, the distance between any two such vertices is always the same and, in fact, equal to $\sqrt{8}$.

Also, there is a symmetry of the hypercube that sends any vertex in $V_{k,n}$ to any other. So, the $C_{k,n}$ are hyperpolyhedra where every vertex has the same configuration of faces emanating from it.

I'm not sure what else I can say that is true in general, so I'll return to working out examples. Since we know what $C_{1,n}$ is, let's try to figure out what $C_{2,n}$ is for general n. Its vertices are the points whose coordinates are all equal to 1 except two coordinates that equal -1, so there are ${}_nC_2 = n(n-1)/2$ vertices. The distance between two such vertices is either $\sqrt{8}$ or 4, depending on whether the coordinates that are equal to -1 overlap. Let P be a vertex in $V_{2,n}$. The vertices in $V_{2,n}$ that are 4 units away from P are the ones where the coordinates that are equal to -1 are completely different from those of P. Therefore, the number of vertices in $V_{2,n}$ that are 4 units away from P is equal to ${}_{n-2}C_2 = (n-2)(n-3)/2$. The other vertices in V_2, other than P, must be $\sqrt{8}$ units from P, so there are ${}_nC_2 - {}_{n-2}C_2 - 1 = 2n - 4$ of those. Unlike the $C_{2,4}$ case, the number of vertices 4 units away is generally greater than the number of vertices that are $\sqrt{8}$ units away. In fact, as n grows, the percentage of vertices that are $\sqrt{8}$ units away tends to 0!

But how are we going to describe the shape $C_{2,n}$?

I guess I'll start with figuring out what its edges are.

In the case of $C_{2,4}$, all the edges were of length $\sqrt{8}$. In fact, any two vertices separated by a distance of $\sqrt{8}$ were the endpoints of an edge. Perhaps the same is true of $C_{2,n}$. Let's find out!

Because $C_{2,n}$ is convex, if E is an edge, then there must be a hyperplane that intersects $C_{2,n}$ exactly in E. Since all the vertices look alike, let's focus on figuring out the edges connected to P. For definiteness, let's take P to have -1's in its first two coordinates (so all its other coordinates are 1). And let's define Q be the vertex with -1's in its first and third coordinates. The smallest coordinates for both of these vertices occur among the first three coordinates, and that makes me think of the hyperplane $x_1 + x_2 + x_3 = -1$. Any vertex that is in this hyperplane *must* have -1's in two of its three first coordinates, because otherwise those coordinates would hold either two or three 1's, resulting in $x_1 + x_2 + x_3 > -1$. That means this hyperplane contains exactly three vertices of $C_{2,n}$:

$$P = (-1, -1, 1, 1, 1, 1, \ldots),\ Q = (-1, 1, -1, 1, 1, 1, \ldots), \text{ and } R = (1, -1, -1, 1, 1, \ldots),$$

and all the other vertices are on one side of it. In fact, this hyperplane intersects $C_{2,n}$ in a triangular face! How nice…instead of finding one edge, we found three, as well as one of $C_{2,n}$'s 2D faces!

By symmetry, any two vertices X and Y that are separated by the shorter distance of $\sqrt{8}$ are indeed the endpoints of an edge of $C_{2,n}$, and each such edge is the side of an equilateral triangle whose third vertex has -1's in the two coordinates where X and Y differ.

In $C_{2,4}$, none of the edges had a length of 4 units. Perhaps that's also true in general. So suppose two vertices have their two -1's in coordinates that do not overlap. Let's say that the two -1's in one of the vertices are in the ith and jth coordinates, and the two -1's in the other vertex are in the kth and lth coordinates, for distinct i, j, k, and l. Suppose that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ is a hyperplane that contains both vertices and all other vertices are on the same side of it.

By substituting the coordinates into the equation for the hyperplane, we see that

$$-a_i - a_j + a_k + a_l = a_i + a_j - a_k - a_l.$$

That is, $a_i + a_j = a_k + a_l$. Without loss of generality, we may assume that substitution of any other vertex's coordinates results in the left-hand side of the hyperplane equation being greater than c. Four other vertices have both their -1's among the coordinates with indices i, j, k, and l. Notice that the sum of the coefficients corresponding to the other indices must be c. Substituting the four other vertices' coordinates into the equation of the plane, we get:

$$\begin{gathered} a_i - a_j + a_k - a_l > 0 \\ a_i - a_j - a_k + a_l > 0 \\ -a_i + a_j - a_k + a_l > 0 \\ -a_i + a_j + a_k - a_l > 0 \end{gathered}$$

If we add the first two inequalities, we get $a_i > a_j$. But if we add the last two inequalities, we get $a_j > a_i$…a contradiction!

That means that no two vertices separated by a distance of 4 are the endpoints of an edge of $C_{2,n}$. (A line segment that joins two vertices 4 units apart *could* be on the boundary of $C_{2,n}$, though, just as the diagonal of a square face of a cube is on the cube's boundary even though it is not an edge.)

That means that we've found *all* the edges of $C_{2,n}$! The edges of $C_{2,n}$ are exactly the line segments connecting pairs of vertices in $V_{2,n}$ that are separated by a distance of $\sqrt{8}$.

Since a line segment connecting a pair of vertices is on the boundary if and only if the length of the segment is $\sqrt{8}$, every facet of $C_{2,n}$ must be equilateral, in the sense that all the edges of the facet must have the same length. This fact gives me hope for finding all the 2D faces, because they would have to be convex equilateral polygons where every pair of vertices is separated by a distance of either $\sqrt{8}$ or 4. This forces the interior angle at any vertex of the polygon to be either 60° or 90°, which means the only possible types of 2D faces are equilateral triangles and squares.

We've already seen that triangular faces exist. What about square faces?

Vertices on opposite ends of a diagonal of the square must be separated by a distance of 4 units. Earlier, we found that the line segment connecting two such vertices cannot be an edge of $C_{2,n}$. But now, we are considering whether it could be on the boundary as the diagonal of a square face. Let's again say that the two -1's in one of the vertices are in the ith and jth coordinates, and the two -1's in the other vertex are in the kth and lth coordinates, for distinct i, j, k, and l. Earlier, we found that any hyperplane that contains the two vertices must have the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ with $a_i + a_j = a_k + a_l$. This time, we don't insist that all other vertices be on one side of the hyperplane. Instead, we require all other vertices to be on one side *or on* the hyperplane, and we require the intersection of the hyperplane with $C_{2,n}$ to be two-dimensional. With a strict inequality, we ran into the contradiction that both $a_i > a_j$ and $a_j > a_i$. Without strictness, we end up with the condition that $a_i = a_j$. By symmetry, we must also have $a_k = a_l$. These facts, combined with $a_i + a_j = a_k + a_l$, show that $a_i = a_j = a_k = a_l$. But then all six vertices whose coordinates have the two -1's among the ith, jth, kth, and lth coordinates would be on the hyperplane. We already know that these six vertices are the vertices of a regular octahedron, which is not a 2D object. This contradiction means square faces do not exist; every 2D face of $C_{2,n}$ is an equilateral triangle!

How many of these equilateral triangular faces are there?

The vertices $\sqrt{8}$ units away from P are the vertices that also have a -1 in one of the first two coordinates. Any two of those that are also $\sqrt{8}$ units apart must form a triangular face with P. Let Q and R (different from the Q and R earlier!) be two vertices $\sqrt{8}$ units away from P. Both Q and R have a -1 as one of their first two coordinates. If the first coordinate of both Q and R is -1, we have a triangular face. If the second coordinate of both Q and R is -1, we have a triangular face. And if the Q and R differ in their first two coordinates, then they must share some coordinate $i > 2$ where they both have a -1. These three cases account for ${}_{n-2}C_2$, ${}_{n-2}C_2$, and $n - 2$ triangular faces, respectively, for a total of $2\,{}_{n-2}C_2 + (n - 2) = (n - 2)^2$ triangular faces.

So, each vertex of $C_{2,n}$ is a vertex of $(n - 2)^2$ triangular faces. Since a triangle has 3 vertices, if we multiply $(n - 2)^2$ by the number of vertices, we will have triple-counted each triangular face. Thus, the total number of 2D faces of $C_{2,n}$ is ${}_nC_2(n - 2)^2/3 = n(n - 1)(n - 2)^2/6$, and they're all equilateral triangles!

Which faces should I try to figure out next?

To be continued…

Learn by Doing

Universal Logic Gates
by Girls' Angle Staff

Note: Problems vary in level of involvement. You'll find that you can do some quickly, but others (in red) are intended to give you something to think about for days. You're always welcome to email us or ask a mentor about this at the club.

In this Learn by Doing, you'll learn about universal logic gates.

But first, what is a logic gate? A logic gate is a function whose input values are all 0 or 1 and whose output value is 0 or 1. Sometimes, people refer to 0 and 1 as "False" and "True," respectively. These two values, 0 and 1, are called **Boolean** values and are named after George Boole, an English mathematician who lived during the 19th century.

As an example, there is the NOT gate, which has one input value. If the input is 0, the output is 1, and if the input is 1, the output is 0. We can summarize the NOT function in a table:

A	NOT A
0	1
1	0

For another example, there is the AND gate, which has two input values. If the input values are A and B, we write the output as A AND B. The AND gate is summarized in this table:

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

In other words, A AND B is 1 if and only if both A and B are equal to 1.

The two tables above that summarize NOT and AND are called **truth tables**.

Another common logic gate is OR. Like AND, OR has two inputs. If the inputs are A and B, the output, A OR B, is 0 if and only if both A and B are equal to 0.

1. Draw the truth table for OR.

We can compose logic gates to form other logic gates.

2. Draw the truth table for NOT (NOT A).

3. Convince yourself that A AND B = NOT ((NOT A) OR (NOT B)).

4. Convince yourself that A OR B = NOT ((NOT A) AND (NOT B)).

The identities in Problems 3 and 4 are known as De Morgan's laws. These laws show that if you have the gates NOT and AND, the gate OR becomes redundant, and, similarly, if you have NOT and OR, then AND becomes redundant.

Often, in colloquial usage of the word "or," people mean "either this or that, but not both." For example, your friend might ask you, "Pizza or ramen?" Usually, your friend means for you to choose one or the other, but not both. But the OR logic gate returns 1 even if both inputs are equal to 1.

The "exclusive or" logic gate, denoted XOR, serves the purpose of providing a logic gate with two inputs that returns 1 if and only if exactly one of the inputs is equal to 1.

5. Draw the truth table for XOR.

6. Express XOR as a composition of the logic gates AND, OR, and NOT.

An example of a logic gate with more than two inputs is "parity 3-bit," which has 3 inputs. If we denote the inputs by A, B, and C, then we'll denote the output by PARITY(A, B, C). The output of parity 3-bit is 1 if and only if the number of inputs equal to 1 is odd. Here's the truth table:

A	B	C	PARITY(A, B, C)
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

7. Express PARITY(A, B, C) as a composition of the logic gates AND, OR, and NOT.

8. Express PARITY(A, B, C) as a composition of just the logic gate XOR.

9. How many different truth tables are there for 2-input logic gates? How many different truth tables are there for 3-input logic gates? For n-input logic gates?

(Spoiler alert!) Did you decide that there are $2^8 = 256$ different truth tables for 3-input logic gates?

10. Show that any 3-input logic gate can be written as a composition of the logic gates AND, OR, and NOT.

11. In fact, show that any logic gate with a finite number of inputs can be expressed as a composition of the logic gates AND, OR, and NOT.

By the way, if you're getting tired of writing AND, OR, and NOT over and over again, these gates are also written "$A \wedge B$" for "A AND B", "$A \vee B$" for "A OR B", and "$\neg A$" for "NOT A".

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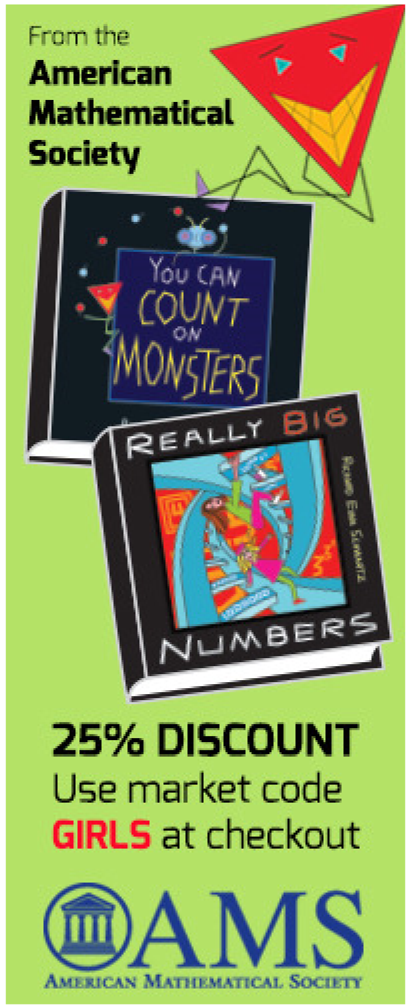


The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

Infinity-Induced Ideas

by Lightning Factorial | edited by Jennifer Sidney

There are a lot of people living on this planet. Yet it is still a finite number of people. If we recorded every person's age at a specific instant in time, there would be a maximum age and a minimum age.

If we instead have an infinitely long list of numbers, the concept of maximum and minimum may no longer apply! For example, suppose we write down the reciprocal of every positive integer: 1/1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, etc. What is the minimum number on this list?

There isn't one!

If you pick any number from the list, there are still smaller numbers to be found on it.

We might wish to call 0 the minimum because it is the largest number that is less than all the numbers on the list. But the problem is that 0 is not one of the numbers on the list! So 0 isn't the minimum, but it *is* an interesting number with respect to this list; wouldn't you agree?

And so we have a new idea induced by infinity: the largest number that is less than all the numbers on the list, dubbed the **infimum** of the list.

Similarly, the smallest number that is greater than every number on the list is called the **supremum**. The supremum of our list of reciprocals is 1. In this case, the supremum is also the list's maximum. In general, as in the situation with infimums, there can be a supremum even when there is no maximum. Can you come up with an example?

The infimum generalizes the concept of minimum to infinite lists of numbers because when the list does have a minimum, it agrees with the infimum; but there can be an infimum without there being a minimum, as our list of reciprocals shows.

Consider the list of numbers sin(1), sin(2), sin(3), sin(4), sin(5), sin(6), …, where we are using units of radians. This list has neither a maximum nor a minimum. But it does have both an infimum and a supremum. (If we were using degrees, then the list *would* have a maximum and a minimum.)

Can you give an example of an infinite list of numbers that not only does not have a maximum or minimum, but also lacks an infimum and a supremum?

Exercise. For each positive integer k, let a_k be a real number. Assume that there exists a real number m such that $a_k > m$ for all positive integers k. Explain why the list of numbers $a_1, a_2, a_3, a_4, \ldots$ has an infimum.

Another infinity-induced idea is the concept of the **limit** of a sequence of numbers. You've probably used this concept already, for example, if you've ever tried to write down the number π (the ratio of the circumference to the diameter of any circle) in decimal notation. You may know that it cannot be expressed as a terminating or repeating decimal. Its decimal representation begins 3.14159… but continues on forever, never falling into a repeating pattern. The more digits we add, the more accurately we represent π, and we can express π to as high a degree of accuracy as we wish by writing down sufficiently many digits. In other words, π is the limit of the sequence of these finite decimal approximations:

In technical terms, if we define the sequence $a_n = \lfloor 10^n \pi \rfloor / 10^n$, then the limit of a_n as n tends to infinity is π.

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

In general, let a_1, a_2, a_3, … be an infinite sequence of numbers. We say that L is the limit of a_k, as k tends to infinity, if and only if for any positive number e,[1] the terms of the sequence are eventually all within e of L; that is, there exists N such that $|a_k - L| < e$ for all $k > N$.

Exercise. Give examples of sequences that do and do not have a limit.

Exercise. Let $f(x) = (x + 2)/(x + 1)$. Define the sequence a_k as follows: Let $a_1 = 1$ and for $n > 1$, let $a_n = f(a_{n-1})$. The first few terms of this sequence are 1, 3/2, 7/5, 17/12, 41/29, … Can you figure out what the limit of the sequence a_k is as k tends to infinity?

The limit of the sequence a_k as k tends to infinity is also written as $\lim_{k\to\infty} a_k$.

Suppose you start solving *The New York Times* crossword puzzle each day. On the kth day, the time it takes you to solve it is T_k seconds. (We're assuming that T_k is the exact time you took, not the time rounded to the nearest second.) The puzzles ramp up in difficulty throughout the week, with Monday being the easiest, and you'd expect this to be reflected in the times T_k. Every now and then, you'll run into an unusually challenging crossword and post a time much longer than your normal time. And sometimes, you'll encounter an unusually easy one and solve it much faster than normal.

Let's suppose that you solve these forever, ending up with an infinite sequence of times T_k.

Say you're interested in getting an idea of the longest it generally takes you to solve these crossword puzzles. If you took the supremum of the sequence, you'd probably get a number that wouldn't accurately reflect what you're looking for. In this case, the supremum would likely also be a maximum, and that maximum might represent the one time that you made a typo and it took you days and days to realize, find, and correct it.

Instead, you want to find a value that excludes such anomalies. For this, we can combine the limit and supremum ideas into the **limit supremum**, which is defined to be the limit of the sequence t_n, where t_n is the supremum of the sequence T_k with its first $n - 1$ terms removed. How would you define the **limit infimum**?

Exercise. If T is the limit supremum of T_k (denoted $\limsup_{k\to\infty} T_k$), show that for any number $S > T$, there are only finitely many terms of the sequence T_k that are greater than S.

Exercise. Let P_k be the kth prime number. What famous conjecture is $\liminf_{k\to\infty} (P_{k+1} - P_k) = 2$?

---

[1] It's become a widely-used convention to utilize the Greek ε (epsilon) where we used "e." It really doesn't matter; you can use any variable you wish.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 36 - Meet 1 January 30, 2025	Mentors:	Elisabeth Bullock, Chloe Kim, Yaqi Li, AnaMaria Perez, Maya Robinson, Dora Woodruff

Some girls began getting a feel for John H. Conway's game **Hackenbush**. In this game, the starting configuration is a collection of line segments joined together at their endpoints and to a horizontal line called the "ground." There can be any number of segments joined at their endpoints. Every segment must either be connected to the ground, or one should be able to reach the ground via a connected string of segments. Each line segment is colored either red or blue. There are two players, one for each color, who alternate turns. On a given turn, a player may erase any segment of her color. Any segments that cannot be traced to the ground are also erased. The last player able to erase a segment is the winner.

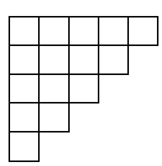
One of the questions that Conway considered and answered is the following: Given a Hackenbush starting configuration H, can a number $N(H)$ be assigned to it such that:

- If $N(H) = 0$, it means that whoever goes first loses.
- If $N(H) > 0$, it means that if the blue player goes first, the blue player wins.
- If H_1 and H_2 are starting configurations, then $N(H_1 + H_2) = N(H_1) + N(H_2)$, where $H_1 + H_2$ is the Hackenbush game whose starting configuration is created by joining the grounds of H_1 and H_2.

Session 36 - Meet 2 February 6, 2025		Meet cancelled due to weather.

Session 36 - Meet 3 February 13, 2025	Mentors:	Elisabeth Bullock, Minerva Johar, Chloe Kim, Hanna Mularczyk, Maya Robinson, Swathi Senthil, Dora Woodruff

How many ways are there to place all the numbers 1 through 15 into the boxes, one number per box, in such a way that down any row or column of boxes, the numbers increase?



Session 36 - Meet 4 February 27, 2025	Mentors:	Elisabeth Bullock, Jade Buckwalter, Sonia Dijkstra, Minerva Johar, Chloe Kim, Shauna Kwag, Jessie Lee, Hanna Mularczyk, AnaMaria Perez, Maya Robinson, Swathi Senthil, Ella Wilson, Dora Woodruff

How would you mathematically model skiing on a mountain?

Calendar

Session 35: (all dates in 2024)

September	12	Start of the thirty-fifth session!
	19	
	26	
October	3	
	10	
	17	
	24	
	31	
November	7	
	14	
	21	
	28	Thanksgiving - No meet
December	5	

Session 36: (all dates in 2025)

January	30	Start of the thirty-sixth session!
February	6	Cancelled - Weather
	13	
	20	No meet
	27	
March	6	
	13	
	20	
	27	No meet
April	3	
	10	
	17	
	24	No meet
May	1	
	8	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

A Math Club for Girls

Girls’ Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls’ Angle Members receive a significant discount on club attendance fees.

Who are the Girls’ Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls’ Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls’ Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members’ efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of “catching up with the group” doesn’t apply.

Where is Girls’ Angle located? Girls’ Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls’ Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls’ Angle activities are tailored to each girl’s specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls’ Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____ Date: _______________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ___________________________________