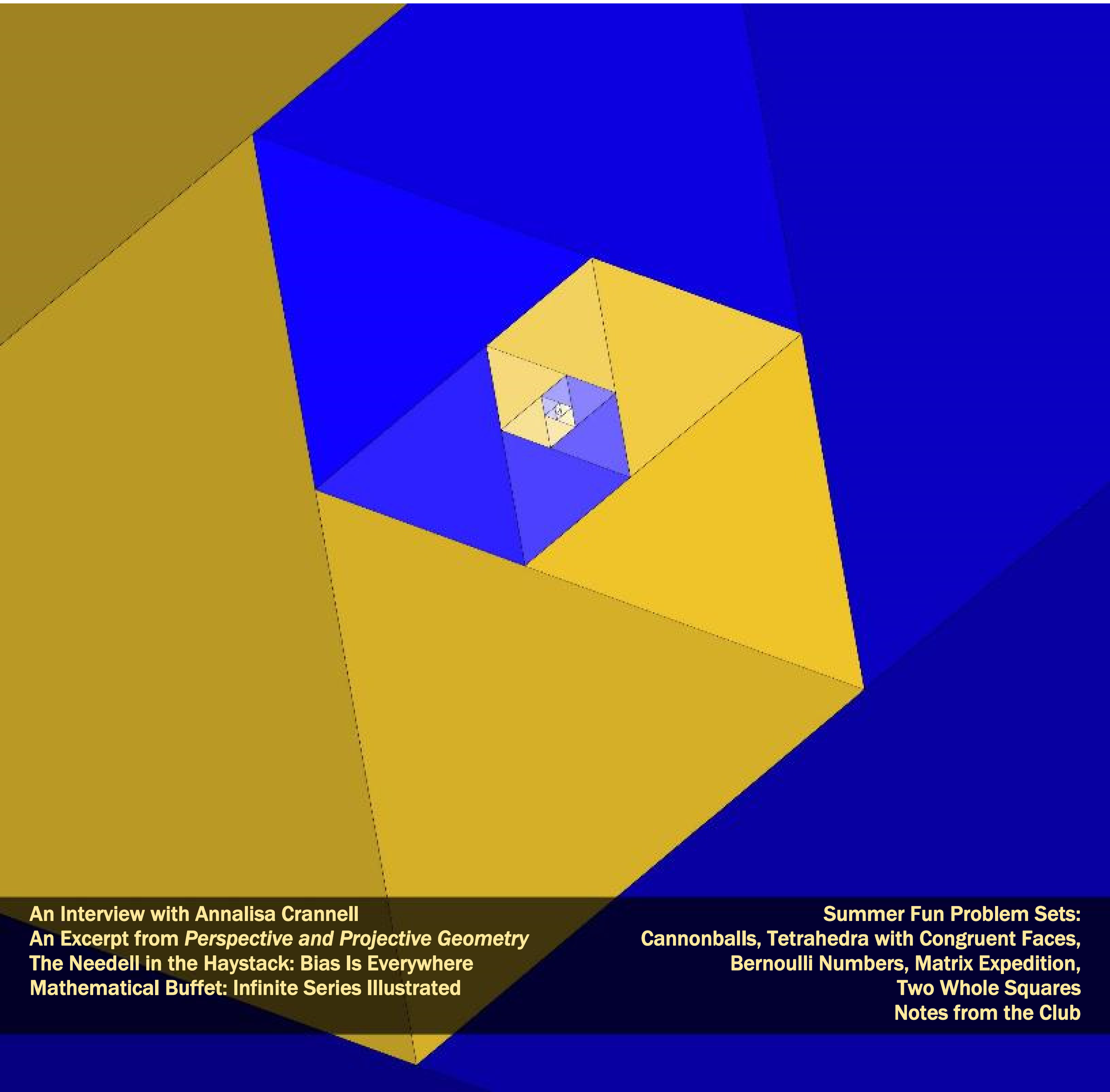
Girls' Angle Bulletin

June/July 2020 • Volume 13 • Number 5

To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Applied math helps us make sense of the world, as Annalisa Crannell shows in her research on perspective drawing. But if improperly applied, math can exaggerate biases, as Deanna Needell explains in her column. So, apply math with great care! - Ken Fan, President and Founder

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Girls' Angle Bulletin

The official magazine of
Girls' Angle: A Math Club for girls
Electronic Version (ISSN 2151-5743)

Website: www.girlsangle.org
Email: girlsangle@gmail.com

This magazine is published six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

The print version of the Bulletin is printed by the American Mathematical Society.

Editor: Jennifer Silva
Executive Editor: C. Kenneth Fan

On the cover: The Fibonacci spiral of triangles as rediscovered by Liliana Smolen and Isabel Wood. See page 15 for more from this bright pair.

An Interview with Annalisa Crannell

Annalisa Crannell is Professor of Mathematics and the Associate Department chair of Computer Science at Franklin and Marshall College. She graduated Magna Cum Laude from Bryn Mawr College and went on to earn her PhD in Mathematics from Brown University under the supervision of Walter Craig. This interview was conducted by Girls' Angle Head Mentor Grace Work.

Annalisa Crannell is the author, together with Marc Frantz and Fumiko Futamura, of "Perspective and Projective Geometry," published by Princeton University Press. Suitable for mathematicians and artists alike, this wonderfully written book combines rigorous mathematics and drawing in a delightfully involving way. You'll find a brief excerpt at the end of this interview.

Grace: What do you love most about mathematics?

Annalisa: This is a hard question, because there are so many things about math that I like. It's kind of like asking, "What do I love most about eating?" Or, "What do I love most about being alive?" But here are some of the kinds of things that make me happy when I'm doing math: having that aha moment when I solve a problem, feeling the flow of being totally focused when I'm working on a tricky problem, sharing some interesting math with other people and seeing their delight, traveling all over the country and giving talks to people who are surprised that math can be so much fun.

Grace: Can you tell us about a time you got really excited about a problem? What was the problem? Why did it excite you?

Annalisa: Here's a fairly recent story. There's this thing in art called "anamorphic distortion". It's when you look at a part of the painting and it looks like just a smear, but when you stand somewhere else and look at painting from an angle, that smear turns into something totally recognizable, like a skull or a portrait. Well, about two years ago I thought I figured out how that was related to a kind of mathematical function called a "homography." I explained it to two of my math BFFs (two people I work with on math a lot) and we ended up writing a paper together using my insight as a jumping-off place, but we didn't quite convince the referees that the anamorphic connection was there, and so I had to take the anamorphic stuff out of the paper.

So we wrote another paper, and I tried to put the anamorphic stuff in, and while we were writing it, my colleagues figured out some other cool stuff, and the paper got so long that we had to take the anamorphic stuff out again. Then a year ago, I was having insomnia because of stress, so I would get up in the middle of the night and work on math, and one night I was playing around on my computer, and I realized the perfect way to describe this result. It was so beautiful, and so simple, and it was unexpected even to me. So we ended up writing a third paper, and we got the anamorphic stuff in. It was so much fun being so certain, and it was really neat to see this one idea turn into so much good math, just like yeast can help turn a little pile of flour into a giant loaf of bread.

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Annalisa Crannell. (We would normally remove the column by Prof. Needell, but her topic addresses bias in data analysis, which is a topic of enormous importance, so we decided to make it available in full.) We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

So, please consider subscribing to the Bulletin. Thanks to our sponsors, subscriptions cost $36/year. With a subscription, you have also gained access to our mentors via email and the ability to influence content in this Bulletin. Visit www.girlsangle.org/page/bulletin_sponsor.html for more information.

Thank you and best wishes,
Ken Fan
President and Founder
Girls' Angle: A Math Club for Girls

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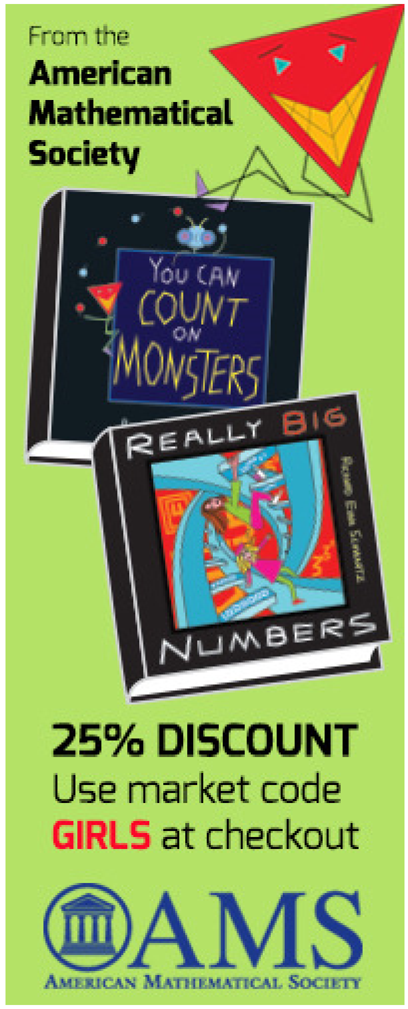

The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

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America's Greatest Math Game: Who Wants to Be a Mathematician.

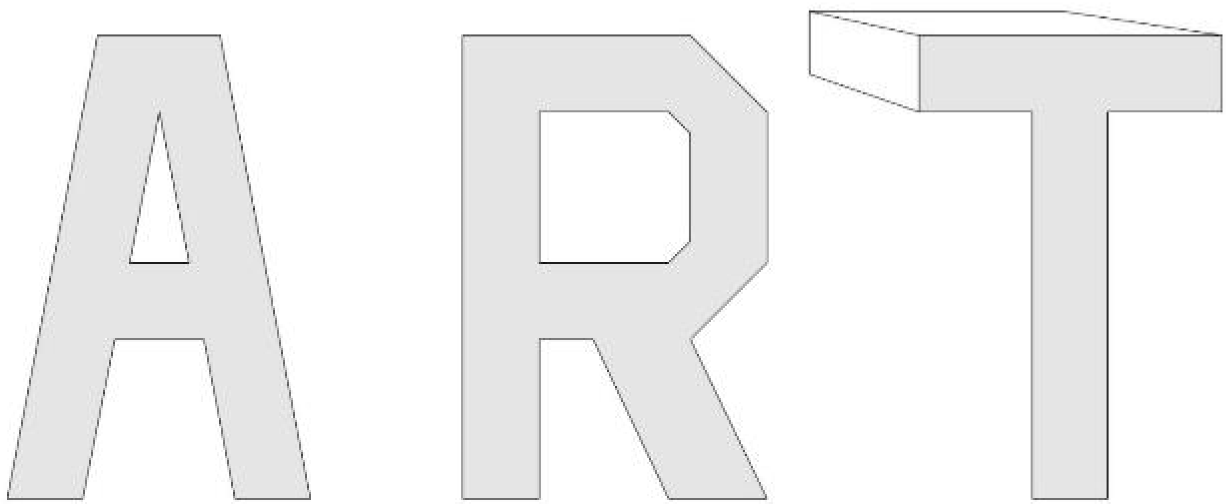


Image from *Perspective and Projective Geometry* reprinted with permission from Princeton University Press

Figure 2.0: The beginning of a drawing of the word ART in one-point perspective.
[For use with the Drawing ART module on the next page.]

relevant attributes (like personality, appearance, speech mannerisms, etc.).

Grace: What do you like doing outside of math? Have you kept up with learning languages?

Annalisa: I do a lot of different things. My husband and I raised six children, a mixture of adopted and birth children. They are grown now. I do volunteer work with a nearby homeless shelter, where sometimes I get to use my Spanish. I love making things with my hands; I have two cordless drills and a sewing machine, and right now I'm busy sewing masks for our local hospital. I care a lot about environmental issues, so for example, this year my husband and I have put only one trashcan at the curb so far (mid-May). We try to reduce our waste. I cook from locally grown food as much as I can, and in fact in the fall my basement is full of food that I canned myself and that we eat all winter until the summer gives us new, fresh fruit and vegetables. I have a bunch of friends who I run with regularly three or four time a week, and six years ago, my husband and I did an Ironman triathlon. So you can see that although math is the center of my life, it's not the only thing in my life by any means!

Grace: What advice or encouragement would you give to students interested in mathematics?

Annalisa: Talk to your teachers! Find a mentor who will encourage you; and after you do that, find another. It's so much more fun to do anything—running, volunteer work, or math—when you've got a group around you who is cheering for you and who enjoys the same kinds of things that you do.

Special thanks to Princeton University Press and Prof. Annalisa Crannell for permission to reprint the excerpt from *Perspective and Projective Geometry* that follows on the next page.

Chapter 2[1]

Drawing **ART**

OVERVIEW:

> In a previous module, we learned that some collections of parallel lines have images that are parallel on the canvas, and other collections of parallel lines have images that pass through a common vanishing point. In this module, we will use what we learned to solve some practical drawing puzzles.

In this module, we will make an 'A' in class. But first we will figure out how to make a T!

Consider Figure 2.1, which shows two attempts to complete a one-point perspective drawing of a 3-dimensional T. That is, the front face of the 3-d T is parallel to the picture plane; lines going into the distance have a vanishing point on the horizon. Both drawings obviously have something wrong with them, even though the back line is vertical and the bottom line goes to the vanishing point.

Figure 2.1: The base of the left-most T is probably too long compared to the top; the base of the right-most T is probably too short.

Your job will be to figure out construction techniques (maybe one, maybe several) for accurately finishing that picture, given a correct start to the picture. See Figure 2.2 on the next page.

1. Drawing directly on Figure 2.2, determine a method for finishing the drawing of the T. (A variety of correct construction techniques are possible, but few of them are immediately obvious. So feel free to take your time, doodle even. Use your straightedge, your pencil, and definitely your eraser!)

[1] Excerpt from *Perspective and Projective Geometry* by Crannell, Frantz, and Futamura, published by Princeton University Press. Published with permission from the author and Princeton University Press.

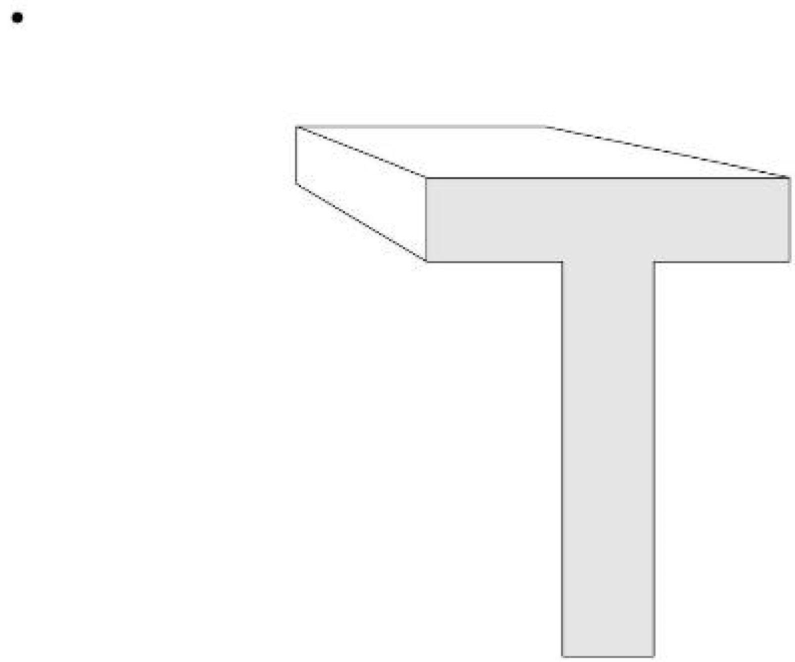


Figure 2.2: Draw the bottom bar of this T.

2. You might discover that your classmates have come up with a variety of clever solutions. On Figure 2.3, sketch several of these for future reference.

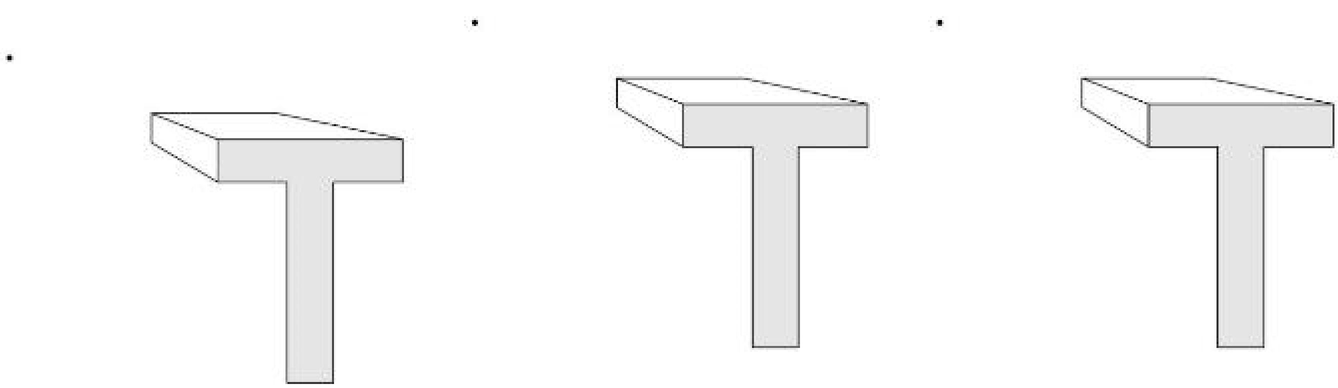


Figure 2.3: Other possible solutions for future reference.

3. Is there one solution that strikes you as particularly "elegant"? If so, what makes that solution appeal to you?

4. What are the advantages/disadvantages of the various solutions?

The Needell in the Haystack[1]

Bias Is Everywhere

by Deanna Needell | edited by Jennifer Sidney Silva

First, I would like to say that I stand with the Black community in demanding change. As a mathematician, data scientist, and educator, I have spent a good deal of time reflecting on my role and how I can best implement change for the better. Like many other sectors, the mathematical community experiences its own dynamics surrounding inclusion, underrepresentation, bias, and more. Clearly, none of these issues are an easy fix, but all progress starts small and stems from passion. I thought it therefore fitting to dedicate this article to a discussion based on bias, and in particular, to comment on a few places bias appears within mathematics and data science. In this context, I will only discuss a very small piece of this problem, covering some of the areas that are most familiar to me. My hope is that if we all do our part to address our own piece, the pieces will someday come together as whole, and we will see more just, brighter tomorrows.

Let's start with the basics: what is bias? The dictionary defines bias in noun form as "prejudice in favor of or against one thing, person, or group compared with another, usually in a way considered to be unfair," and as a verb, "cause to feel or show inclination or prejudice for or against someone or something." As a mathematician, I often hear my colleagues in other fields tell me how fortunate I am that I work in a field with no subjectivity. The perception is often that mathematics, by its nature, is completely objective and void of opinion, interpretation, or evaluative judgement. Therefore, one may conclude that it is also free of bias; after all, $2 + 2 = 4$, and there is no one who will dispute that (we will save that objection for another day!). However, bias appears in our discipline in many ways. As already mentioned, it appears in the representation and involvement of the mathematical community. It appears in who has access to textbooks, journals, and other mathematical resources. It appears in every way bias appears in society at large, as the mathematical community is a subset of it. But today I want to focus on how bias appears in the mathematics itself. In applied mathematics in particular, this is a growing concern, and one that is important both societally and mathematically.

I will focus on two main sources of bias in mathematics: bias that comes from the data itself, and bias that stems from the mathematical tools used to analyze that data. I hope that these discussions serve as a springboard into studying these issues, as it would be impossible to provide a full account of the problem in one article.

Let's begin with a well-known example. Suppose Sheila and May are in a friendly competition about who is the best at bird-watching. In the first month, Sheila goes bird-watching on 18 different days, and on 16 of those days she spots a bird. In the same month, May goes bird-watching twice and sees a bird both times. In the second month, Sheila goes twice and sees a bird on 1 of those days, whereas May goes 18 times and spots a bird 10 times. Who is the better bird-watcher? The problem is summarized in the tables below.

Month 1

	Sheila	May
Successes	16	2
No Bird	2	0
TOTAL	18	2

Month 2

	Sheila	May
Successes	1	10
No Bird	1	8
TOTAL	2	18

[1] This content supported in part by a grant from MathWorks.

Let's crunch some numbers. In the first month, Sheila spotted a bird 16/18, or about 89% of the time. On the other hand, May spotted a bird every time (2/2), so she was successful 100% of the time. And in the second month, Sheila only spotted a bird 50% of the time (1/2), whereas May did 56% (10/18) of the time. So in the first month May was more often successful, and in the second month May was also more successful! You might be about to throw a party for May, or you might be feeling that something is awry. Let's try another computation. In both months combined, Sheila went bird-watching 18 + 2 = 20 times, and spotted a bird on 16 + 1 = 17 occasions. Her success rate is thus 17/20 = 85%. May, on the other hand, also went a total of 2 + 18 = 20 times but was successful 10 + 2 = 12 times, giving her a success rate of 12/20 = 60%. So Sheila is actually the winner. Wait a second, what happened here?

What we are witnessing is a simple but illustrative example of what is known as Simpson's paradox. Although not really a paradox, it is named as such because of the seemingly contradictory nature of the predicament. At its heart is the simple observation that one can construct numbers a, b, c, d, e, f, g, h such that both

$$\frac{a}{b} > \frac{c}{d} \text{ and } \frac{e}{f} > \frac{g}{h},$$

but

$$\frac{a+e}{b+f} < \frac{c+g}{d+h}.$$

The implication of this observation is that the way in which the data is *aggregated* can significantly affect the interpretation of what the data represents. Indeed, when the bird-watching data was analyzed per month, each month showed May to be the clear winner. However, when it was aggregated over both months combined, Sheila won. This example of bird-watching is, of course, quite silly; however, Simpson's paradox has appeared in a wide array of more important settings, including a now well-known example involving admissions to UC Berkeley. The relevant admissions data is given in the following table (departments have been anonymized. Source: https://en.wikipedia.org/wiki/Simpson%27s_paradox.

Department	**Men**		**Women**	
	Applicants	**Admitted**	**Applicants**	**Admitted**
A	825	62%	108	**82%**
B	560	63%	25	**68%**
C	325	**37%**	593	34%
D	417	33%	375	**35%**
E	191	**28%**	393	24%
F	373	6%	341	**7%**

Looking at the data in this format, would you assess there to be any bias by UC Berkeley? Most likely, as the university did at the time, one would argue that there is not a major source of discrimination here. Indeed, Departments A, B, D, and F admitted higher percentages of women than men, and Departments C and E, while admitting higher percentages of men than women, yielded percentages that only differ by at most 4% between men and women. This data did not alarm the university, and the analysis concluded that no course of action needed to be taken. However, let us aggregate the data over all departments. In this case, the results appear as follows:

	Men		Women	
	Applicants	**Admitted**	**Applicants**	**Admitted**
Total	8442	**44%**	4321	35%

When viewed as aggregated data, it appears that a much higher percentage of men were accepted than women. The point I make here is not to argue which analysis is "right" or "wrong," but to highlight that discrepancies exist that can be quite drastic when analyzing data in different ways. This example has now been analyzed extensively, and many argue that because more women applied to more competitive departments (Departments C and E, for example), this amplified the effect. Simpson's paradox and this issue appear in any type of data that can be categorized into natural groups, including clinical trial data from various patient groups, longitudinal data that is grouped into time slots, and other data grouped by demographics.

The moral of this story, in my opinion, is that bias can appear in many forms even within simple data analyses. We need to take extraordinary care when drawing conclusions from data, and be transparent in how those conclusions were drawn.

The second – and perhaps more obvious – source of bias comes from the data itself. There is bias within the collection of data, so the sources and reliability of such data should always be questioned. Here, though, I focus on data that – although reliable in and of itself – may have been sourced from a population in a biased way. I should clarify that this does not imply that the collector of the data did anything neglectful or intentional, as she often has access only to limited data. Our question then is how to best handle this imposed bias. Let me start with an example that ties back to the problem of nonnegative matrix factorization (NMF) that we have seen in several prior articles.

Recall that the NMF problem seeks a factorization of a d by n data matrix X into a product WH, where W is a d by r matrix with non-negative entries representing the r learned **dictionary atoms** and H is the non-negative r by n **code matrix**. Each column of the data matrix X is represented by a (non-negative) linear combination of the columns of the dictionary matrix with coefficients given by the corresponding column of the code matrix. By viewing these two matrices together, one essentially learns r "topics" that explain the data contained in X.

Since my last article talked about some of our recent involvement in Covid-19 work, I will use this data as an example. This work is joint with Jana Gevertz, Rachel Grotheer, Jamie Haddock, Alona Kryshchenko, Hanbaek Lyu, and Chris Strohmeier. There is an abundance of literature on the coronavirus (not just the recent novel coronavirus), including over 17,000 (English) articles. One can apply NMF to the problem of organizing this enormous library of articles by creating a data matrix X whose rows correspond to articles and columns correspond to word counts for words appearing in those articles. From the dictionary matrix, the words corresponding to the largest entries can be extracted (e.g., the 15 words with highest values in each topic) and used to interpret the revealed topics.

Table 1 below shows an example output from this dataset using 10 topics. The center column shows the 15 words with highest entries for each topic, and the rightmost column is a possible interpretation from those words. Notice that in this case, there is not always an obvious interpretation when looking at the words alone. Additionally, though, the words listed in those topics themselves depend on the input data, which in this case is the word counts for all of the articles used. One simple source of bias in this example is the number of words used in each document. Indeed, documents that are simply longer (and thus contain more words in general) will have higher impact on these topics than shorter articles. This bias occurs not because the data itself was collected in any sort of inappropriate fashion, but simply because the data that is accessible inherently has variability in terms of content and size. Similar types of bias occur in other settings, such as those in which groups of people do not have equitable representation in the sample.

There are many methods that are used to attempt to balance the effect of this bias, the simplest of which is to use random downsampling or upsampling. For example, the dataset X can be normalized to have the same word count in each row by randomly selecting the same number of words from each document. In our case, doing a drastic downsampling by only using the *title* of each article rather than the full text results in Table 2, below. There appears to be a difference in the topics identified. This work is ongoing, so we are far from drawing conclusions yet. However, it already highlights the importance of taking care throughout the data analysis process, from the collection of data to applying methods and interpreting results.

Topic Number	Key Words	Possible Topic
0	respiratory children infections patients rsv tract pneumonia infection viruses acute cause clinical viral age common	Viral Respiratory Infections in Children
1	et al 2014 2013 2012 2015 2016 2011 2010 2017 2008 2006 2005 2007 2009	Dates in References (not helpful)
2	cov mers sars coronavirus respiratory east middle syndrome human saudi coronaviruses arabia severe 2012 camels	MERS Outbreak
3	rna proteins viral protein viruses virus genome host replication membrane structural sequence gene species dna	Microbiology/Gene Sequencing
4	influenza pandemic virus h1n1 viruses h5n1 avian 2009 human seasonal h7n9 flu pandemics poultry humans	Related/Past Pandemics (other virus types)
5	health diseases public disease infectious data transmission emerging surveillance control outbreaks population global outbreak countries	Public Health Response
6	a1111111111 dengue zikv denv transmission host emergence endemic virus asia infection aerosols disease pathogen prrsv	Past/Related Virus Transmission
7	pedv diarrhea porcine ped pigs piglets swine virus epidemic disease industry tgev vomiting losses coronaviridae	Related Coronaviridae with Porcine Origins
8	cells cell immune ifn infection response responses expression il inflammatory innate receptors type activation cancer	Cellular Immunology
9	2019 china covid wuhan cases 2020 ncov sars patients 19 coronavirus confirmed january pneumonia preprint	Initial Outbreak in China

Table 1. The results of nonnegative matrix factorization applied to a dataset consisting of words from a library of articles related to the current pandemic.

Topic Number	Key Words	Possible Topic
0	virus infection hepatitis ebola replication zika dengue host syncytial bronchitis entry fever antiviral activity feline	Related Viral Diseases
1	syndrome middle east respiratory coronavirus mers severe korea cov acute outbreak 2015 saudi reproductive camels	Previous Coronavirus-Caused Diseases
2	respiratory infections viral acute children tract patients severe infection hospitalized clinical study pneumonia bacterial human	Respiratory Symptoms
3	coronavirus sars novel china 2019 covid cov 19 clinical wuhan patients transmission ncov mers outbreak	Original Covid-19 Outbreak in China
4	protein human cells cell expression viral rna host immune response induced type responses interferon receptor	Molecular Immunology
5	porcine epidemic diarrhea genome virus strain sequence complete china characterization pedv analysis piglets reproductive pigs	Related Virus Strain Originating in Pigs
6	influenza pandemic h1n1 2009 avian h5n1 h7n9 transmission risk surveillance illness study like hemagglutinin factors	Related/Past Pandemics (other virus types)
7	infectious diseases emerging disease bronchitis vaccines review bmc control surveillance research tropical microbes www molecular	Prevention and Surveillance
8	health public care research china global emergency review ministry bmc access study surveillance department workers	Public Health Response
9	viruses detection rna pcr time real using assay rapid based development amplification isothermal reverse loop	Time Sensitive Detection

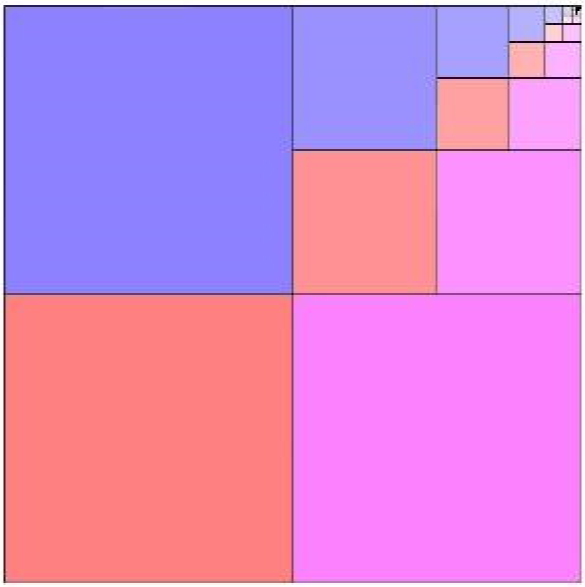
Table 2. The results of nonnegative matrix factorization applied to a dataset consisting of words from the titles of the articles whose words were used to produce Table 1.

Knowing that bias is introduced at every stage, acknowledging this, and taking it into account both methodologically and in conclusions is absolutely critical. Handling bias in data and data science is an aspect of mathematical research that is important and continues to grow.

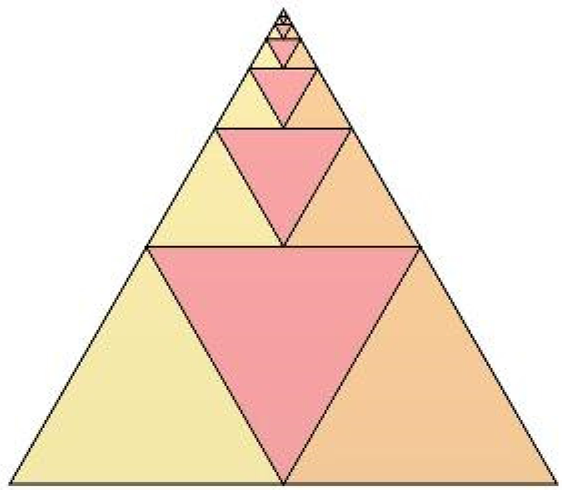
Mathematical Buffet: Infinite Series Illustrated

by Liliana Smolen and Isabel Wood

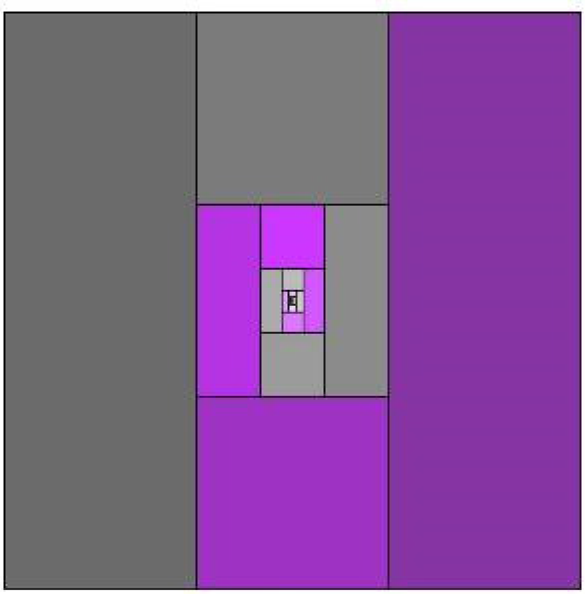
Images that enable one to see a mathematical truth just by looking are always interesting. There are many in the book *Proofs without Words: Exercises in Visual Thinking* by Roger Nelsen. Here and on the cover, we present five that Liliana and Isabel dreamt up in an hour of mathematical play. Although these particular ideas aren't new discoveries, Liliana and Isabel created them for themselves completely from scratch. - Editor



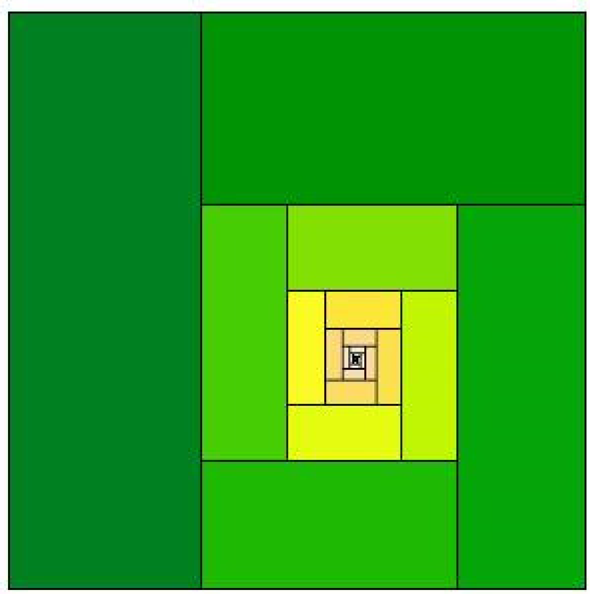
$$\frac{1}{3}=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$$



$$\frac{1}{3}=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\ldots$$



$$\frac{1}{2}=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$$



$$1=\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\frac{16}{243}+\ldots$$

Summer Fun!

The best way to learn math is to do math. Here are the 2020 Summer Fun problem sets.

We invite all members and subscribers to send any questions and solutions to us at girlsangle@gmail.com. We'll give you feedback and might put your solutions in the Bulletin!

The goal may be the lake, but who knows what wonders you'll discover along the way?

In the August issue, we will provide some solutions. You could wait until the August issue to see the answers, but you will learn a lot more if you try to solve these problems on your own.

Some problems are very challenging and could take several weeks to solve, so please don't approach these problems with the idea that you must solve them all. Our main goal is to give you some interesting things to think about.

If you get stuck, try to formulate a related question that you can see a way to actively explore to get your mind moving and your pencil writing. If you don't understand a question, email us.

If you're used to solving problems fast, it can be frustrating to work on problems that take weeks to solve. Try to enjoy the journey and don't be afraid to follow detours. It's like hiking up a mountain. Getting to the top rewards one with a spectacular view, but during the journey, there's a lot to see and experience. So here's a meta-problem for those of you who feel frustrated when doing these problems: see if you can dissolve that frustration and replace it with a relaxed, optimistic sense of adventure!

This is Summer Fun, not Summer Torture!

Summer Fun!

Cannonballs and Combinatorics

by Annie Yun

We define ${}_nC_k$, which is read "n choose k," to be the number of ways you can pick out a set of k items from a set of n items. Here, the order in which the items are picked does not matter. For example, how many 3-elements subsets of the set {1, 2, 3, 4, 5} are there? We can systematically list them:

{1, 2, 3}	{1, 2, 4}	{1, 2, 5}	{1, 3, 4}	{1, 3, 5}
{1, 4, 5}	{2, 3, 4}	{2, 3, 5}	{2, 4, 5}	{3, 4, 5}

We have just found that ${}_5C_3 = 10$. For reasons which will become clear, we define ${}_nC_0 = 1$ for all $n \geq 0$. We also define ${}_nC_k = 0$ if $k > n$.

1. How many ways are there to select 2 apples from 10 apples? (These 10 apples are each unique.) In other words, what is ${}_{10}C_2$? How many ways are there to select 2 apples from n apples? In other words, find a formula for ${}_nC_2$.

2. Generalize your formula for ${}_nC_2$ to a formula for ${}_nC_k$ for any nonnegative integers n and k. (For an answer, see answer A on page 29.)

3. Here are the top eight rows of Pascal's famous triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

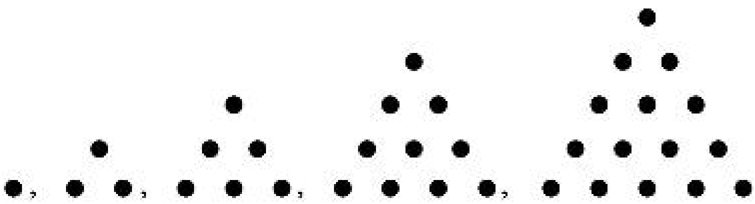
By convention, we refer to the top row as row 0 so that row n will begin 1, n, …. Starting with a single 1 in the top row, each further entry is the sum of the one or two numbers directly above it (more precisely, directly above left and directly above right). For example, 21 = 6 + 15.

What does this triangle of numbers have to do with ${}_nC_k$? Prove whatever you think to be true. (For example, prove that the formula you found in Problem 2 satisfies the rules for generating Pascal's triangle, namely ${}_0C_0 = 1$ and ${}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$ for all $n > 0$ and $0 \leq k \leq n$.)

4. Give an algebraic proof that ${}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$ for all $n > 0$ and $0 \leq k \leq n$. Now prove the same equation "bijectively," that is, pair up each k-element subset of the set $\{1, 2, 3, \ldots, n + 1\}$ with a different $k - 1$- or k-element subset of the set $\{1, 2, 3, \ldots, n\}$, and vice versa.

5. Spend some time finding patterns in Pascal's triangle and proving that the patterns you see aren't mere coincidences. (For example, just about every book on combinatorics will ask you to show that the sum of the entries in any row of Pascal's triangle is a power of 2.) There are many, many patterns in Pascal's triangle!

6. The triangular numbers are the numbers that count the number of dots in a triangular arrangement of dots:



Each successive triangle can be formed by adding a new row of dots below, where the new row has one more dot than the previous bottom row. We obtain a sequence that begins 1, 3, 6, 10, …. Show that the nth term in this sequence is ${}_{n+1}C_2$.

7. When can a perfect square of dots be rearranged to form a triangular arrangement of dots as in Problem 6? In essence, when is ${}_nC_2$ a perfect square?

If you're stuck on Problem 7, here are a few hints. Skip to Problem 8 if you love the challenge.

7a. You have to find integer solutions to the equation $n(n-1)/2 = m^2$. This is equivalent to the equation $(2n-1)^2 - 2(2m)^2 = 1$. The equation $x^2 - 2y^2 = 1$ is a particular **Pell equation**. Pell's equations are equations of the form $x^2 - py^2 = 1$, where p is a positive, non-square, integer.

7b. Notice that if x and y are big whole umbers that satisfy $x^2 - 2y^2 = 1$, then x/y is a good rational approximation to $\sqrt{2}$. One way to generate good rational approximations to $\sqrt{2}$ is to look at the partial fractions of its continued fraction expansion. That is, observe that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}},$$

then look at the fractions you get by truncating this infinite fraction: 1 = 1/1, 1 + 1/2 = 3/2, 1 + 1/(2 + 1/2) = 7/5, etc. Note that the numerators and denominators to every other fraction you obtain in this way, starting with 3/2, provides a solution to $x^2 - 2y^2 = 1$.

7c. Another way to find solutions to $x^2 - 2y^2 = 1$ is to define x_k and y_k by the equation $x_k + y_k\sqrt{2} = (3+2\sqrt{2})^k$, for $k \geq 0$. Show that $x_k^2 - 2y_k^2 = 1$, for $k \geq 0$. Can you prove that these pairs (x_k, y_k) give *all* the solutions to the Pell equation $x^2 - 2y^2 = 1$?

The border of Pascal's triangle consists entirely of 1's. One in from the border, we find numbers of the form ${}_nC_1 = n$ and ${}_nC_{n-1} = n$, and among these numbers, all integers greater than 1 appear twice. In particular, in this border one in from the edge, every perfect square greater than 1 appears twice. In Problem 7, you found all the perfect squares that are in the border of Pascal's triangle two in from the edge.

8. Build a square pyramid out of cannonballs in the following way: Start with a square base that consists of 9 cannonballs in a 3 by 3 square arrangement. On top of this layer, place 4 cannonballs in a 2 by 2 square arrangement so that each rests upon four cannonballs in the first layer. Finally, top off the pyramid with a cannonball that rests on the four cannonballs in the second layer. Generalize this construction to n layers. Define the nth square pyramidal number S_n to be the number of cannonballs required to build such a square pyramid with n layers. (So $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$.) Show that $S_n = {}_{n+2}C_3 + {}_{n+1}C_3 = n(n+1)(2n+1)/6$.

9. In 1875, Edouard Lucas asked, "When can a perfect square number of cannonballs be arranged into a square pyramid?" In other words, when is S_n a perfect square? That is, find positive integer solutions to the equation

$$\frac{n(n+1)(2n+1)}{6} = m^2 .$$

By multiplying this equation by 4 and making appropriate substitutions, show that this equation is equivalent to finding even numbers a and b such that ${}_aC_3 = b^2$.

10. Problem 9 motivates the question, "What perfect squares are there in the border of Pascal's triangle three in from the edge?" Lucas correctly claimed that there is only one solution. Can you find it? (Also, see the Summer Fun problem set "Bernoulli Numbers" on page 22.)

11. Lucas's proof of his claim in Problem 10 was flawed. It took decades before a proof was found, and decades more before proofs were found that only involved basic number theory. So if you're up for a really big challenge, try to prove Lucas's claim. For more on the history and an elementary proof, we refer the reader to "The Square Pyramid Puzzle," by W. S. Anglin on pages 120-124 of *The American Mathematical Monthly*, Volume 97, 1990.

12. In fact, there are no other perfect squares in Pascal's triangle. This is a special case of a theorem of Paul Erdős, who, in 1951, showed that there are no solutions in integers m and n to the equation ${}_nC_k = m^l$ for any $l \geq 2$ and $4 \leq k \leq n - 4$. We refer the interested reader to a presentation of his proof in *Proofs from THE BOOK*, by Aigner and Ziegler, pages 15-18.

Tetrahedra with Congruent Faces

by Ken Fan

The regular tetrahedron is a triangular pyramid whose faces are congruent equilateral triangles. It is the Platonic solid with the fewest number of faces.

1. How many vertices, edges, and faces does a regular tetrahedron have?

2. Let s be the side length of a regular tetrahedron. Show that its surface area is $\sqrt{3}s^2$ and its volume is $\sqrt{2}s^3/12$. (Recall that the volume of a pyramid is $Bh/3$, where B is the area of the base and h is the height of the pyramid with respect to that base.)

An arbitrary tetrahedron can have 6 edges with 6 different lengths. There are formulas for the surface area and volume of a tetrahedron in terms of its edge lengths, and if you're looking for an algebraically intense challenge, deriving these formulas fits the bill. Instead, we'll look at an intermediary class of tetrahedra which slightly generalizes the regular tetrahedron: tetrahedra whose four faces are all congruent to each other. Such tetrahedra have many names, among them **isosceles tetrahedra** and **disphenoids**.

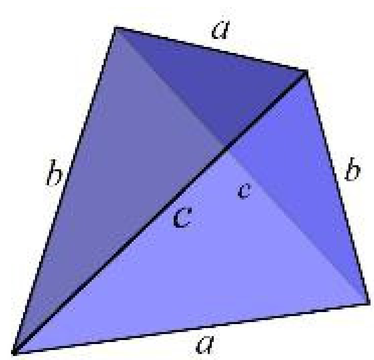
Let T be an isosceles tetrahedron. Let a, b, and c be the side lengths of one (or any one) of its faces, labeled so that $a \le b \le c$.

3. Show that the faces are acute triangles, which can be expressed algebraically by

$$c^2 < a^2 + b^2.$$

Two edges of an icosahedron are said to be **opposites** if they do not share an endpoint.

4. Show that opposite edges of an isosceles tetrahedron have the same length.



Since all four faces of T are congruent, its surface area is four times the area of any one of its faces. So finding a formula for the surface area of T in terms of a, b, and c is tantamount to finding the area of a triangle in terms of its side lengths.

5. Express the area of a triangle with side lengths a, b, and c in terms of its side lengths. (The formula you'll find is known as Heron's formula.)

6. Check that 4 times the formula you found in Problem 5 reduces to $\sqrt{3}s^2$ when $s = a = b = c$.

Summer Fun!

7. Show that the volume of T is given by

$$\sqrt{\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}{72}}.$$

8. Check that the formula in Problem 7 reduces to $\sqrt{2}s^3/12$ when $s = a = b = c$.

9. In a face of T, let A be the angle opposite the side of length a, let B be the angle opposite the side of length b, and let C be the angle opposite the side of length c. Show that the volume of T can also be expressed as

$$\frac{abc}{3}\sqrt{\cos A \cos B \cos C}.$$

Every tetrahedron can be inscribed in a sphere, called the circumscribed sphere. That is, there is always a unique sphere which passes through all four vertices of the tetrahedron.

10. Show that the diameter of the circumscribed sphere of T is given by

$$\sqrt{\frac{a^2+b^2+c^2}{2}}.$$

Every tetrahedron contains an inscribed sphere, which is a sphere contained inside the tetrahedron and is tangent to all four faces of the tetrahedron.

11. Show that the radius of the inscribed sphere of T is given by $3V/S$, where V is the volume of T and S is its surface area.

12. Show that the inscribed and circumscribed sphere of T are concentric.

13. Show that the center of the inscribed and circumscribed spheres of T is the centroid of T.

14. Let m_a be the line segment that joins the midpoints of the edges of T of length a. Similarly, define m_b and m_c. Show that the lengths of m_a, m_b, and m_c are given by the formulas

$$\sqrt{\frac{b^2+c^2-a^2}{2}},\ \sqrt{\frac{c^2+a^2-b^2}{2}},\text{ and }\sqrt{\frac{a^2+b^2-c^2}{2}},$$

respectively.

15. Prove that m_a, m_b, and m_c (from Problem 14) are mutually perpendicular.

Summer Fun!

Bernoulli Numbers

by Matthew de Courcy-Ireland

Define a sequence of numbers B^n by $B^0 = 1$ and, recursively,

$$(B - 1)^n = B^n, \text{ for } n \neq 1.$$

This is to be interpreted by expanding $(B - 1)^n$, but then interpreting the exponents as superscripts in the sequence. For example, the definition entails $B^1 = 1/2$ because the recursion relation for $n = 2$ says that $(B - 1)^2 = B^2$. The expansion of $(B - 1)^2$ is $B^2 - 2B^1 + B^0$, so

$$B^2 - 2B^1 + B^0 = B^2.$$

Interpreting the exponents as superscripts, this simplifies to $B^1 = B^0/2$, and we already know that $B^0 = 1$. Thus, $B^1 = 1/2$.

> Remember, in this Summer Fun problem set B^2 and B^3 are not "B squared" and "B cubed"! Instead, they are the second and third terms of a special sequence of numbers. Throughout this problem set, a superscript that directly decorates a capital B stands for an index and not an exponent!

1. Verify that $B^2 = 1/6$, $B^3 = 0$, and $B^4 = -1/30$. Compute several more terms.

2. Let n and p be positive integers. Show that

$$(B + n)^{p+1} = (B + n - 1)^{p+1} + (p + 1)n^p.$$

Remember: The way to interpret this equation is that after expansion, B^k is to be regarded not as "B to the kth power," but instead as term k of the sequence B^n. For example, when $p = 1$, we are to show that

$$(B + n)^2 = (B + n - 1)^2 + 2n.$$

For the left hand side of this equation, we have

$$(B + n)^2 = B^2 + 2nB^1 + n^2B^0 = 1/6 + 2n(1/2) + n^2(1) = 1/6 + n + n^2.$$

For the right hand side, we have

$$(B + n - 1)^2 + 2n = B^2 + 2(n - 1)B^1 + (n - 1)^2B^0 + 2n = 1/6 + 2(n - 1)(1/2) + (n - 1)^2 + 2n.$$

Since $1/6 + n + n^2 = 1/6 + 2(n - 1)(1/2) + (n - 1)^2 + 2n$, the case $p = 1$ is verified.

3. Deduce that

$$(B + n)^{p+1} = B^{p+1} + (p + 1)(1^p + 2^p + 3^p + \ldots + n^p).$$

4. Express the sums

$$1 + 2 + 3 + \ldots + n$$

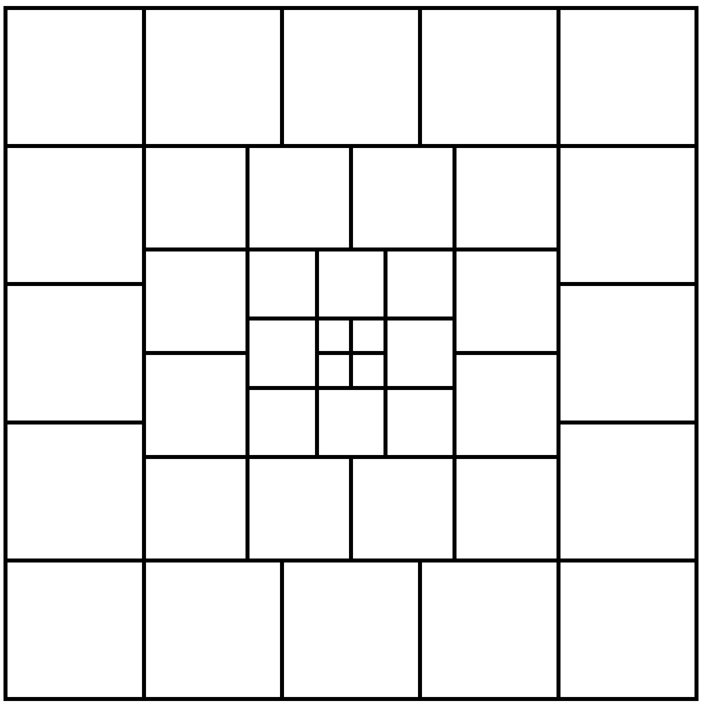
$$1^2 + 2^2 + 3^2 + \ldots + n^2$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3$$

as polynomials in n.

5. Give a geometric interpretation of your formula for $1 + 2 + 3 + \ldots + n$.

Can you explain geometrically how this sum is related to $1^3 + 2^3 + 3^3 + \ldots + n^3$? Here's a hint for one way to do this:



6. Can you find any examples where the sum of the first n squares is itself a perfect square? (Also, see the Summer Fun problem set "Cannonballs and Combinatorics" on page 17.)

7. Can you express $\cos(Bx)$ and $\sin(Bx)$ as simple functions of x? What does this tell you about B^n with n odd? Here, $\cos(Bx)$ and $\sin(Bx)$ are defined by interpreting $(Bx)^n$ as $B^n x^n$ in their series expansions. What about other functions, like e^{Bx}?

8. How quickly can you compute $1^{10} + 2^{10} + 3^{10} + \ldots + 1000^{10}$?

Summer Fun!

Matrix Expedition

by Jasmine Zou

This Summer Fun problem set is intended for people who are familiar with vectors and matrices. For those who are unfamiliar, take a look at any book on linear algebra or see the Learn By Doing *on pages 8-11 of Volume 7, Number 3 of this Bulletin.*

In this Summer Fun problem set, all of our vectors will be position vectors in the *xy*-coordinate plane. That is, all vectors will be thought of as arrows pointing from the origin (0, 0) to some point (x, y) in the plane. This vector will be denoted $\begin{pmatrix} x \\ y \end{pmatrix}$. Matrices will then be regarded as linear transformations of the coordinate plane. If you are unfamiliar with vectors and matrices, we should caution that one should not generally think of a vector as an arrow pointing from the origin to a point in some coordinate system. In general, a vector should not be thought of as having its foot tied to some location in space, nor are coordinates required to speak of them.

Refresher: if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are matrices and $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is a vector, then

$$Mv = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \quad \text{and} \quad MN = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

1. What matrix corresponds to a reflection in the vertical axis?

2. What matrix corresponds to a reflection in the line $y = x$?

3. What matrix corresponds to a rotation about the origin by 180°?

4. Let t be an angle measure. What transformation of the plane is effected by the matrix

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}?$$

5. What matrix sends every vector to itself? Let's call this matrix **1**. Show that $M\mathbf{1} = \mathbf{1}M = M$ for any matrix M.

6. Find all matrices that send the vector (2, 3) to the vector (3, 2).

7. Find all matrices that send vectors that point (from the origin) to a point on the line $y = 2$ to vectors that point (from the origin) to a point on the line $y = 1 - x$.

Summer Fun!

8. Find two matrices M and N such that MN is *not* equal to NM.

9. Find all matrices M that satisfy $M^2 = \mathbf{1}$.

10. Find a matrix E such that $E^3 = \mathbf{1}$. (See Problem 5 for the definition of **1**.)

11. Use matrices to deduce the trigonometric angle sum formulas

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

and

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

12. Let F_n be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n > 0$. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Express M^n in terms of the Fibonacci sequence. Use determinants to figure out the value of $F_{n+1}F_{n-1} - F_n^2$.

13. Evaluate $\displaystyle\sum_{n=1}^{\infty} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 1/3 \end{pmatrix}^n$.

14. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the transpose of M, denoted M^T, is defined by $M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. In other words, M^T is the matrix M flipped over its northwest-southeast diagonal. Let

$$F = \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 1/3 \end{pmatrix}.$$

Determine $F + FF^T + FF^TF + FF^TFF^T + FF^TFF^TF + \ldots$.

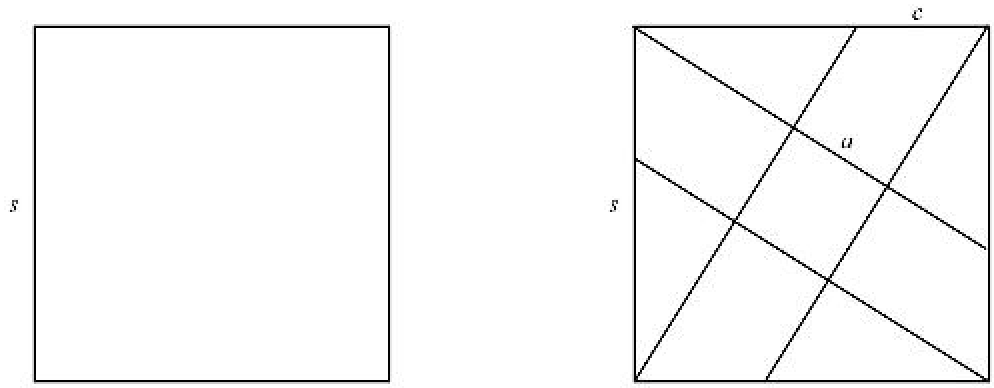
15. What is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$?

16. Let M be a matrix and suppose that $M^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for some positive integer n.

Prove that $M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Two Whole Squares

by Ken Fan and Grace Work



We start with a square of side length s. We then draw lines from each corner to a point c away from the opposite corner (measuring in the counterclockwise direction around the square), as shown in the diagram above right. Assume that $0 \leq c \leq s$. These four lines form a smaller square inside the original. We call its side length a.

1. Convince yourself that the four slanted lines do indeed form the sides of a square.

2. Find an expression for a in terms of s and c.

3. Suppose we desire that both squares have integer side lengths. Show that if s is a whole number, then it can be arranged that a have any whole number value less than or equal to s.

As a practical matter, if you actually want to make this design, it would be really convenient if not only s and a are whole numbers, but if also c is a whole number. It's a lot easier to measure off a whole number of centimeters than it is to measure off, say, $\sqrt{2}$ centimeters. So the next problem asks you to find solutions to the equation you found in Problem 2 where s, c, and a are all whole numbers. If you want hints, the ensuing problems break the problem into steps. But if you want to try to find all the solutions on your own, do Problem 4, and then skip to Problem 15.

4. Determine all whole number solutions to the equation you found in Problem 2.

If $c = s$, then the four slanted segments won't be slanted at all; they'll coincide with the sides of the original square, and $a = s$. If $c = 0$, then the slanted segments will fall upon the diagonals of the square and $a = 0$. Let's avoid these special cases and assume that $0 < c < s$.

Notice that all relationships between lengths and angles in the figure remain intact under scaling, so if s, c, and a are a solution, then so is xs, xc, and xa. Therefore, it makes sense to look for integer solutions s, c, and a that share no common factor other than 1. Let's call such a solution **primitive**. All other solutions can be obtained by scaling a primitive solution.

Summer Fun!

5. Show that $c > a$.

This means that we can make a right triangle with hypotenuse of length c and one leg of length a. Let b be the length of the other leg. That is, b is defined so that $c^2 = a^2 + b^2$.

6. Show that $b/a = (s - c)/s$.

7. Show that b must be a whole number. (If you're stuck, see hint B on page 29.)

8. Show that a, b, and c are a primitive Pythagorean triple. (Recall that we are seeking the solutions where s, c, and a share no common factor other than 1.)

All primitive Pythagorean triples can be obtained by taking two relatively prime, positive integers m and n, not both odd, with $m > n$. Then $m^2 - n^2$ and $2mn$ are the lengths of the legs of a primitive Pythagorean right triangle with hypotenuse $m^2 + n^2$. For example, if $m = 2$ and $n = 1$, these formulas yield the 3, 4, 5 right triangle.

9. Rearrange the equation in Problem 6 to see that $s = ca/(a - b)$.

The equation $s = ca/(a - b)$ implies that $a > b$. Therefore, we can find relatively prime, positive integers m and n, with $m > n$, and not both odd so that a is the larger of $m^2 - n^2$ and $2mn$, and b is the smaller, and $c = m^2 + n^2$.

10. Show that a and $a - b$ are relatively prime. Show that c and $a - b$ are relatively prime.

11. Explain why $a - b$ must be 1.

12. Observe that $(m^2 - n^2) - 2mn = (m - n)^2 - 2n^2$.

It is not clear which of $m^2 - n^2$ and $2mn$ is the larger number, but Problems 11 and 12 enable us to conclude that $(m - n)^2 - 2n^2$ must be either +1 or -1.

13. If we let $x = m - n$ and $y = n$, then either $x^2 - 2y^2 = 1$ or $x^2 - 2y^2 = -1$. These are examples of Pell equations. Define x_k and y_k by the equation $x_k + y_k\sqrt{2} = (1 + \sqrt{2})^k$ for $k \geq 0$. Show that

$$x_k^2 - 2y_k^2 = (-1)^k .$$

14. Use Problem 13 to explain how to find all the primitive triples s, c, and a.

15. For a similar challenge, find all triples s, c, and a of positive integers where s is the side length of an equilateral triangle, and an equilateral triangle of side length a is formed by drawing line segments from each vertex to a point c units along the opposite side as measured in the counterclockwise direction.

Summer Fun!

Notes from the Club

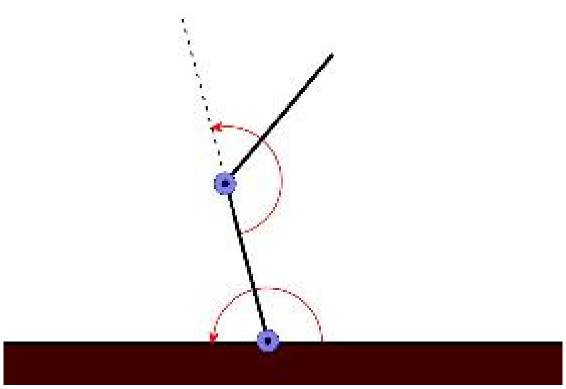
These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 26 - Meet 12 April 30, 2020 Online	Mentors:	Rebecca Nelson, Kate Pearce, Laura Pierson, Savannah Tynan, Emma Wang, Rebecca Whitman, Hanna Yang, Jasmine Zou

We engaged some members with math games that work well in the Zoom meeting application. Actually, one of the games was a game we often play and refer to as the "Describe the Shape" game. One person is given a design and attempts to get others to replicate it by using only spoken words. The game forces participants to identify patterns in the design and communicate precisely. Another game was Math Show-and-Tell. In this game, the mentor states a word or concept, such as "cube" or "1/2" and members then go off to find something in their house that illustrates the concept, then returns to show the others and explain the connection.

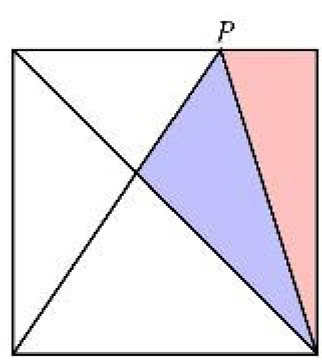
Session 26 - Meet 13 May 7, 2019 Online	Mentors:	Tina Lu, Rebecca Nelson, Kate Pearce, Laura Pierson, Gisela Redondo, Emma Wang, Rebecca Whitman, Hanna Yang, Jasmine Zou

We held our traditional end-of-session Math Collaboration, modified for the online setting, this one designed by Rebecca Nelson and Head Mentor Grace Work, who continue a tradition of mentors creating spectacular events that began with Alexandra Fehnel in Session 21, followed by Jacqueline Garrahan, Elise McCormack, Jenny Kaufman, and Laura Pierson. A huge Thank You to them all! Here are a couple of problems from the event.



A robot arm is composed of 2 segments, each a meter long. These segments are connected together, and to the wall, by joints with a range of 180° as indicated in the figure at left. At the free end is a paintbrush. Calculate the area of the region that can be painted by this robot arm.

At right is a unit square. Point P is situated on the upper side so that the areas of the blue and red regions are equal. How far is P from the upper right corner of the square?



Calendar

Session 26: (all dates in 2020)

January	30	Start of the twenty-sixth session!
February	6	
	13	
	20	No meet
	27	
March	5	
	12	Meet cancelled
	19	Girls' Angle club meets are now conducted online
	26	
April	2	
	9	
	16	
	23	
	30	
May	7	

Session 27: (all dates in 2020)

September	10	Start of the twenty-seventh session!
	17	
	24	
October	1	
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	

Due to current circumstances, Girls' Angle club meets are being conducted online until further notice. We look forward to when we can return to our in-person club.

Here are some answers and hints to Summer Fun problems.

A. ${}_nC_k = n!/(k!(n-k)!) = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1}$.

B. Hint: If the square root of a whole number is rational, then in fact the square root must be a whole number.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____