Girls' Angle Bulletin

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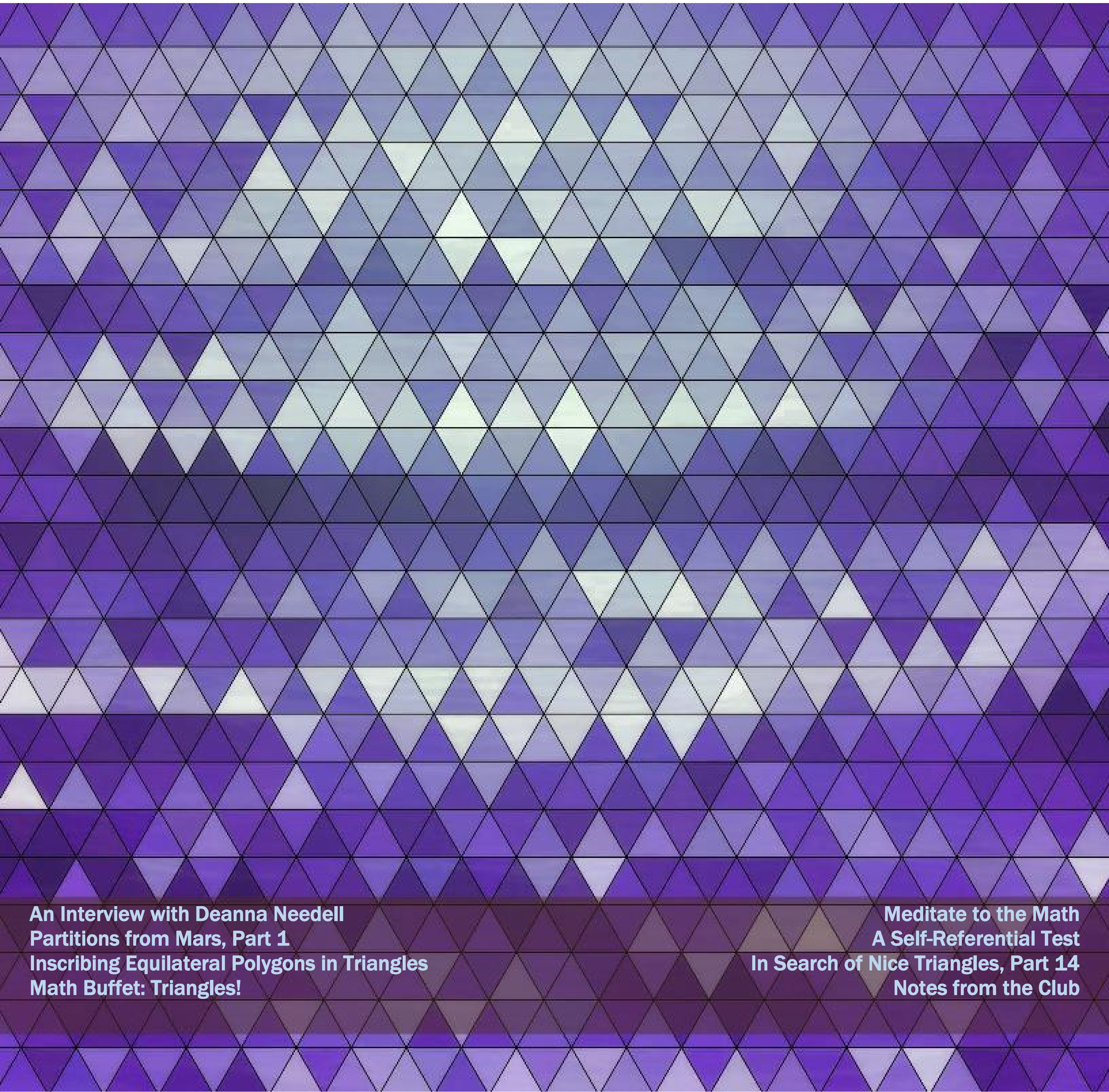
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Take three points. If they don't fit inside a line, you've got yourself the vertices of a triangle. It is such a basic shape, yet it opens the door to so much math, some of which can be found in the pages of this issue.

- Ken Fan, President and Founder

Girls' Angle Bulletin

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Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Just a Few Triangles* by C. Kenneth Fan.

An Interview with Deanna Needell

Deanna Needell is professor of mathematics at UCLA. Her research interests include compressed sensing, randomized algorithms, functional analysis, computational mathematics, probability, and statistics.

Ken: What is compressed sensing?

Deanna: Compressed sensing is a growing topic in mathematics, computer science, statistics, and engineering that proposes novel techniques for signal processing and compression. For example, you may wish to store an image that is of size 1000 by 1000 pixels, making up 1,000,000 pixels in total. This would take up a lot of memory if each of those 1,000,000 pixel values had to be stored. Instead, compressed sensing suggest a particular kind of "sampling" allows one to store far fewer values than 1,000,000 while still being able to reconstruct the image accurately from those stored values. The mathematics behind such techniques include methods in random matrix theory, numerical and real analysis, geometry, and more. It requires the design not only of sampling methods but also sophisticated algorithms to do the reconstruction accurately and efficiently.

Ken: What fascinates you about compressed sensing?

Deanna: I like being able to use "pure" tools from mathematics to solve real world problems. I enjoy working on theory and writing proofs, but also being able to impact the world in a positive way. I feel areas of applied math like compressed sensing allow me to do that. In addition, we develop many large-scale data analysis techniques that apply in high dimensions, so we are able to use surprising and beautiful results from high dimensional geometry, while also

contributing to the growing important area of big data science.

Ken: One of your achievements was to help doctors do MRI scans of patients more effectively. Could you please explain how your work does that, and can you give us a sense of how and how much MRI scans can be improved using your techniques?

Deanna: Medical imaging, and in particular MRI, is an important and practical application of compressed sensing. Just like one can "sample" an image as described above, MRI uses samples in what is called "frequency space." A typical MRI may take upwards of 40 minutes to complete, because the machine needs to measure many of these samples. Using tools from compressed sensing, we can select a smaller number of these samples and still reconstruct an accurate image of whatever is being scanned. This allows to significantly reduce the scan time needed. This is especially critical in pediatric MRI where the patient may not be able to hold perfectly still for very long.

Ken: Does the brain implement compressed sensing algorithms?

Deanna: This is the question, for sure! Scientists do not of course know how the

Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free.

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For this issue, those who do not subscribe to the print version will be missing out on the remainder of this interview with Prof. Deanna Needell and some other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
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President and Founder
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Partitions from Mars, Part 1[1]

by Pamela E. Harris, Alexander Pankhurst, Cielo Perez, and Aesha Siddiqui[2]
and edited by Jennifer Silva

In memory of Bertram Kostant (May 24, 1928 – February 2, 2017)

If you have been to an arcade and exchanged tickets for prizes, you've encountered an integer partition function problem. For example, if you won 17 tickets and a piece of gum is 1 ticket, an eraser is 5 tickets, and a pencil is 10 tickets, then you could exchange all 17 of your tickets in the six ways depicted in Table 1.

Pencils	**Erasers**	**Gum**
0	0	17
0	1	12
0	2	7
0	3	2
1	0	7
1	1	2

Table 1. Ticket Exchanges.

The problem we consider here is a similar one, except we are at an arcade on Mars where they don't use tickets. Instead, the currency comes in two different types: α_1 and α_2. To purchase an item on Mars, you need the right amount of α_1 and the right amount of α_2. For example, a solar system marble set costs $1\alpha_1$ and $1\alpha_2$. One very important peculiarity about martian currency is that there is no amount of α_1's that we can exchange for any amount of α_2's.

Martian currency can be conveniently modeled by the concept of a **vector**, which is a mathematical quantity with more than one piece of information. In particular, our aforementioned martian currency is modeled by the vectors

$$\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

These vectors can be added/subtracted together by adding/subtracting the entries in the same positions. Multiplication can be performed by multiplying each entry in the vector by a specified multiplier. We say two vectors are equal only when all of their entries are equal. Let's illustrate these operations with an example:

$$3\alpha_1 + 6\alpha_2 = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+0 \\ 0+6 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}.$$

Notice that the peculiar property of martian currency – namely, that no amount of α_1's can be exchanged for any amount of α_2's – is respected by using vectors. We can see that regardless of what number we multiply the vector α_1 by, we will never be able to make the second entry (from top to bottom) equal anything other than 0, whereas any amount of α_2's would have a nonzero second entry. By considering the top entries of the vectors α_1 and α_2, we can also deduce that there is no number of α_2's that we can exchange for α_1. This means that α_1 and α_2 cannot be exchanged for one another. But alas, that is how currency works on Mars!

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[1] This content is made possible by a grant from MathWorks.
[2] All authors are affiliated with the Department of Mathematics and Statistics at Williams College.

Now let's suppose that at the martian arcade we won $3\alpha_1$ and $6\alpha_2$ worth of martian currency, which we express as $3\alpha_1 + 6\alpha_2$. The arcade items that we can exchange our martian currency for are as follows:

Item	Martian Price
Eclipse sunglasses	α_1
Deimos rocks	α_2
Solar system marble set	$\alpha_1 + \alpha_2$

Notice that when we write $\alpha_1 + \alpha_2$, it means we must use one each of α_1 and α_2 in order to purchase a solar system marble set.

Now that we understand martian currency, the question we want to answer is this: In how many ways could we exchange $3\alpha_1 + 6\alpha_2$ in martian currency for the above arcade items? To do a correct count let's figure out how many solar system marble sets we could purchase. If we don't buy any solar system marble sets, then we can get 3 eclipse sunglasses and 6 Deimos rocks. If we instead purchase 1 solar system marble set, then with the remaining martian currency ($2\alpha_1 + 5\alpha_2$) we can purchase 2 eclipse sunglasses and 5 Deimos rocks. We also could purchase 2 solar system marble sets, and with the remaining martian currency ($\alpha_1 + 4\alpha_2$) we could get a pair of eclipse sunglasses and 4 Deimos rocks. Lastly, we could purchase 3 solar system marble sets and with the remaining martian currency ($3\alpha_2$) we could purchase 3 Deimos rocks. This gives a total of four different ways we could exchange $3\alpha_1 + 6\alpha_2$ for the arcade items.

When dealing with these counting problems, we let $\wp(m\alpha_1 + n\alpha_2)$ indicate the number of ways we can exchange $m\alpha_1 + n\alpha_2$ in martian currency for eclipse sunglasses, Deimos rocks, and solar system marble sets, at the prices given above; each way to do so is called a **vector partition**. In our example above, we have determined that $\wp(3\alpha_1 + 6\alpha_2) = 4$. We now want to determine the total number of ways to exchange $m\alpha_1 + n\alpha_2$ in martian currency for the arcade prizes when m and n are nonnegative integers. This is our first result.

Lemma 1. If m and n are nonnegative integers, then $\wp(m\alpha_1 + n\alpha_2) = \min(m, n) + 1$.

Proof. As we saw in the example, the answer to the problem depends only on counting the number of solar system marble sets we purchase. Suppose that we buy s solar system marble sets. Then we have $(m - s)\alpha_1 + (n - s)\alpha_2$ in martian currency remaining, which can be exchanged in exactly one way: $m - s$ pairs of eclipse sunglasses and $n - s$ Deimos rocks. Notice that the smallest number of solar system marble sets we can purchase is 0, while the largest number of solar system marble sets we can purchase depends completely on whether m or n is smaller. So we have determined that $0 \le s \le \min(m, n)$. Thus, there are $\min(m, n) + 1$ ways to exchange $m\alpha_1 + n\alpha_2$ of martian currency for the arcade items. □

Let's now consider how the problem changes when we have the additional martian currency vector

$$\alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

As before, we cannot exchange any number of α_1's and α_2's for α_3. This is because no matter what numbers we multiply the vectors α_1 or α_2 by, we will never be able to get a vector with a nonzero bottom entry, whereas any nonzero amount of α_3 has a nonzero bottom entry. In fact, no sums of α_1's and α_2's will ever give an α_3 because such sums will also have 0 in the bottom entry. Likewise, we cannot exchange any number of α_3's for α_1 or α_2. So once again, the martian currencies α_1, α_2, and α_3 are not interchangeable.

Suppose that in this new setting we won $4\alpha_1$, α_2, and $3\alpha_3$ worth of martian currency, which we denote by $4\alpha_1 + \alpha_2 + 3\alpha_3$. The arcade items we can now purchase with our martian currency are:

Item	Martian Price
Eclipse sunglasses	α_1
Deimos rocks	α_2
Astronaut bobblehead	α_3
Solar system marble set	$\alpha_1 + \alpha_2$
Pocket telescope	$\alpha_2 + \alpha_3$
Antigravity socks	$\alpha_1 + \alpha_2 + \alpha_3$

In how many ways can we exchange $4\alpha_1 + \alpha_2 + 3\alpha_3$ in martian currency for the arcade items above? Table 2 depicts all of the possibilities.

Item	Eclipse sunglasses	Deimos rocks	Astronaut bobblehead	Solar system marble set	Pocket telescope	Antigravity socks
Price per item	α_1	α_2	α_3	$\alpha_1 + \alpha_2$	$\alpha_2 + \alpha_3$	$\alpha_1 + \alpha_2 + \alpha_3$
Number of items purchased	4	1	3	0	0	0
	3	0	3	1	0	0
	4	0	2	0	1	0
	3	0	2	0	0	1

Table 2. Exchanging martian currency for arcade prizes.

As before, we want to determine the total number of ways to exchange $m\alpha_1 + n\alpha_2 + k\alpha_3$ in martian currency for the arcade prizes, when m, n, and k are nonnegative integers. To begin, we assume that m, n, k satisfy the condition $m, k \geq n$. We now state our next result.

Proposition 1. If m, n, and k are nonnegative integers with $m, k \geq n$, then

$$\wp(m\alpha_1 + n\alpha_2 + k\alpha_3) = (n + 1)(n + 2)(n + 3)/6.$$

Proof. Our proof of this result can be framed as a balls-in-urns problem. Recall that if you have N identical balls and R urns, there are ${}_{N+R-1}C_{R-1}$ ways to distribute the N balls among the R urns. (Here, ${}_nC_k$ denotes the binomial coefficient "n choose k," which is the coefficient of x^k in the expansion of $(1 + x)^n$.) In our setting, we observe that once we know how many Deimos rocks, solar system marbles, pocket telescopes, and antigravity socks are purchased, then the number of eclipse sunglasses and astronaut bobbleheads we can get (with the remainder of our martian currency) is uniquely determined. This is because α_2 is involved in the price of all items except for the eclipse sunglasses and the astronaut bobbleheads. Now we can think of n, the coefficient of α_2, as the number of balls we can distribute into the four bins labeled by the costs of the Deimos rocks (α_2), solar system marbles ($\alpha_1 + \alpha_2$), pocket telescopes ($\alpha_2 + \alpha_3$), and antigravity socks ($\alpha_1 + \alpha_2 + \alpha_3$). Since $m, k \geq n$, we know that any distribution of our n α_2's into these four bins corresponds to a realizable purchase of items. That is, since $m, k \geq n$, after purchasing the items that involve α_2 in the cost, we will still have a positive amount of α_1 and α_3 left to purchase eclipse sunglasses and astronaut bobbleheads. So there are ${}_{n+3}C_3$ distinct ways to exchange $m\alpha_1 + n\alpha_2 + k\alpha_3$ in martian currency for the arcade items. Note that

$$ {}_{n+3}C_3 = (n + 3)(n + 2)(n + 1)/6. \qquad \square $$

The first few numbers generated by the formula in Proposition 1 are

1, 4, 10, 20, 35, 56, 84, 120, 165, 220….

These numbers are known as **tetrahedral numbers**, and Figure 1 provides a way to visualize them, as the manner in which we would stack spheres in a triangular pyramid.

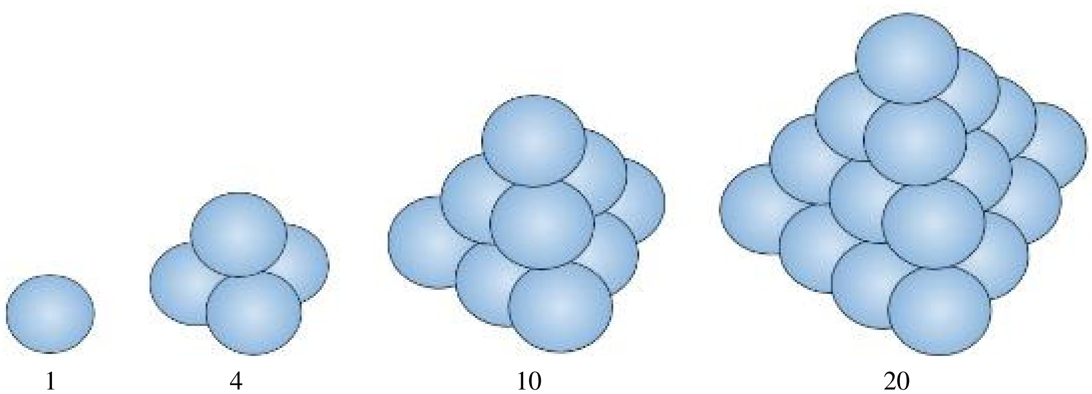

Figure 1. The first few tetrahedral numbers illustrated.

Now let's try to compute $\wp(m\alpha_1 + n\alpha_2 + k\alpha_3)$ under the condition $m, n \geq k$ by using the balls-in-urns method. In this case, we have k balls and three urns labeled by α_3, $\alpha_2 + \alpha_3$, and $\alpha_1 + \alpha_2 + \alpha_3$. The balls-in-urns method would tell us that there are ${}_{k+2}C_2$ distinct ways to distribute our k α_3's among the three items that involve α_3 in their cost. However, each such distribution does not uniquely determine a way to finish exchanging the remainder of our martian currency since we will, in general, be able to purchase different numbers of, say, solar system marble sets with what we have left. For example, suppose we start with $4\alpha_1 + 3\alpha_2 + 2\alpha_3$ in martian currency. One way we can distribute the two α_3's among the three items that involve α_3 in their cost is to spend both α_3's to purchase 2 astronaut bobbleheads, which cost α_3 each. This would leave us with $4\alpha_1 + 3\alpha_2$ to spend on eclipse sunglasses, Deimos rocks, and solar system marble sets, and lemma 1 informs us that there are four different ways to do that.

This means we cannot employ a direct balls-in-urns approach to compute the value of $\wp(m\alpha_1 + n\alpha_2 + k\alpha_3)$. If we tried to, for each distribution of the k α_3's into three urns, we'd have to count how many ways there are to complete the shopping cart to expend the rest of the martian money, then add up all these tallies. Stay tuned for Partitions from Mars, Part 2 for the general solution to this problem!

Concluding Remarks

The type of problem we investigated above is part of an active field of mathematical research on **vector partition functions**. These functions play important roles in chemistry and physics, as scientists are interested in partitioning a number of particles (or molecules) in a system based on the available energy levels [1]. For example, if we start with *N* particles and there are *m* available energy levels, then the partition function would account for a way to distribute the N particles among these levels. This is similar to the balls-in-urns problem, but often there are restrictions as to how many particles can be in a specific energy level.

The specific vector partition function that we are using as an example in this article is known as the vector partition for the "positive roots of the Lie[3] algebra $sl_4(C)$," also referred to as the type A_3 Lie algebra. Lie algebras are named after Sophus Lie, a Norwegian mathematician from the late 1800's who is credited with creating the theory of continuous symmetry [2]. The partition function we studied is an example of a **Kostant partition function**, named after Bertram Kostant, an American mathematician who was a leader in the fields of representation theory and Lie theory. His recent passing motivated our work on this article, and we dedicate the manuscript to his memory.

Kostant's partition function plays a role in determining the dimensions for certain special vector spaces called weight spaces. These dimensions are also referred to as weight multiplicities, and in the 1950's Kostant connected this work with his now well-known weight multiplicity formula. This formula reduced the multiplicity problem to a sum whose terms involve counting the number of ways we can express a vector as a sum of certain other vectors [3]. Hence, his work provided a new way to compute these multiplicities via a vector partition function approach! Many mathematicians continue to work in this very active field of research. Indeed, the first author of this article devoted her Ph.D. thesis to the study of counting problems related to Kostant's weight multiplicity [4]; in Part 2 of this article we will discuss further progress in this study.

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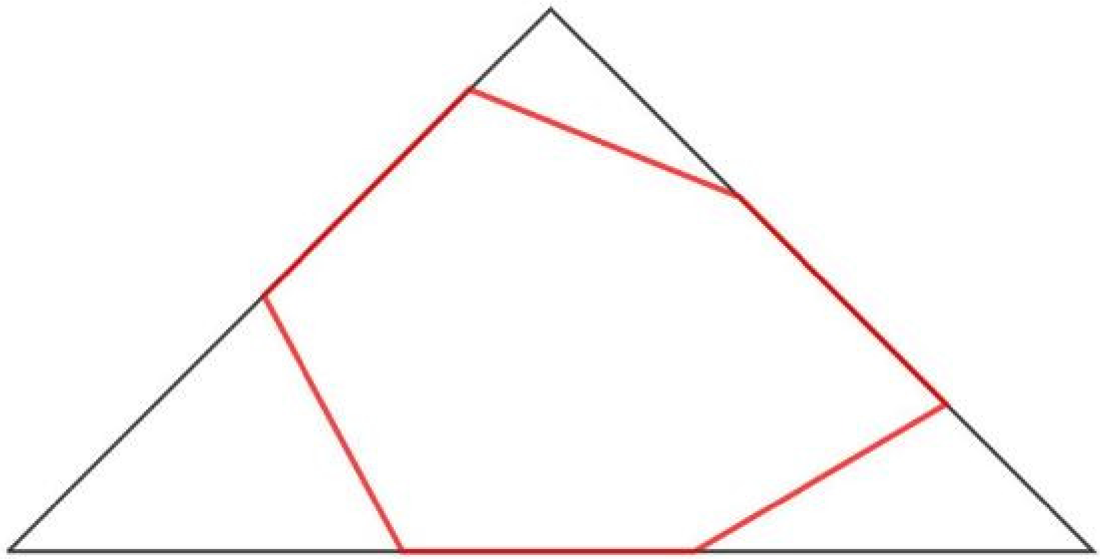
[3] Here, "Lie" is pronounced "Lee" and not like the word "lie."

Inscribing Equilateral Polygons in Triangles

by Milena Harned and Miriam Rittenberg | edited by Jennifer Silva



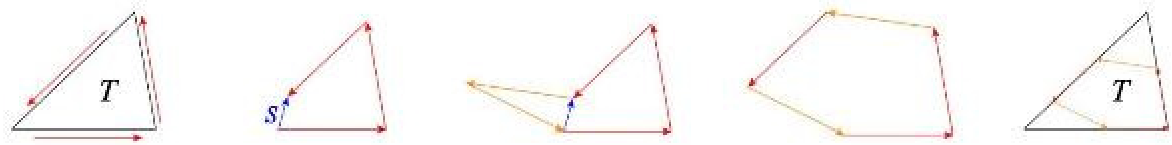
Introduction

Given a triangle, is it possible to inscribe a convex equilateral n-sided polygon such that three of the polygon's sides are flush with the sides of the triangle (see figure above)?

In this article, we'll show that this is impossible if $n < 5$, except for the isolated case of inscribing an equilateral triangle in itself. If $n > 5$, there are an infinitely many solutions. We'll discuss the cases $n = 5$ and $n = 6$ in detail.

Strategy

Let T be a triangle. For our purposes, it's convenient to think of the triangle's sides as having a direction, oriented so that the sides point counterclockwise around the triangle. Finding an equilateral n-gon inscribed in T such that three of the n-gon's sides are flush with the sides of the triangle is equivalent to finding n unit vectors that sum to zero, such that three of the vectors point in the direction of the sides of the triangle. We can build the n-gon by starting with the three unit vectors that point in the direction of the sides of the triangle. Suppose these three vectors have vector sum S. We then find $n – 3$ unit vectors whose sum is $-S$. Once we have these n vectors, we can form a unique (up to congruence) equilateral convex n-gon by placing the vectors end to end in counterclockwise order of their directions. We can then scale the n-gon as necessary so that it fits snugly in T. An example of this process when $n = 5$ is shown in Figure 2 at the top of the next page.



We start with 3 unit vectors in the directions of the sides of T.

Let S be the sum of these unit vectors.

We find 2 unit vectors that add up to $-S$.

We add up the 5 unit vectors in counterclockwise order of their directions.

We then scale the resulting pentagon so that it fits snugly into triangle T.

Figure 2. Illustrating our strategy with $n = 5$.

In order to find unit vectors that add up to $-S$, we need the following lemma:

Lemma: Let T be a triangle. Let u, v, and w be unit vectors that point in the direction of the sides of the triangle, where the sides are oriented to point around the triangle in a counterclockwise direction. Then $|u + v + w| < 1$.

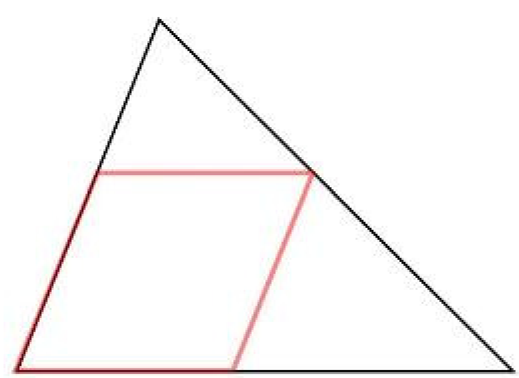
Proof: Let triangle ABC be similar to T, labeled so that BC corresponds to the longest side of T and scaled so that $BC = 1$. By the triangle inequality, $BA + CA > BC = 1$. Extend BA beyond A to P if necessary, so that $PB = 1$. Extend CA beyond A to Q if necessary, so that $QC = 1$. We then have an illustration of the vector sum $u + v + w$ and $PQ = |u + v + w|$. By the triangle inequality, $PQ < PA + QA = (1 - BA) + (1 - CA) = 2 - (BA + BC) < 1$.

Triangles

We can pick three unit vectors in the direction of the sides of a triangle T; if the vectors do not sum to zero, we cannot add any more vectors, so we cannot inscribe a triangle in T. If the vectors do sum to zero, then they form an equilateral triangle; furthermore, this triangle must be similar to T because it has the same angles as T, so T must also be equilateral.

Quadrilaterals

Equilateral quadrilaterals are rhombi, which are all parallelograms. Thus, we cannot put three sides of a rhombus along three of a triangle's sides since a triangle cannot have parallel sides. If we instead allow the rhombus to have only two sides flush with sides of the triangle, then there are three solutions, one for each vertex. Each solution shares one of the vertices, and its side length is half the harmonic mean of the lengths of the sides of the triangle that meet at that shared vertex.



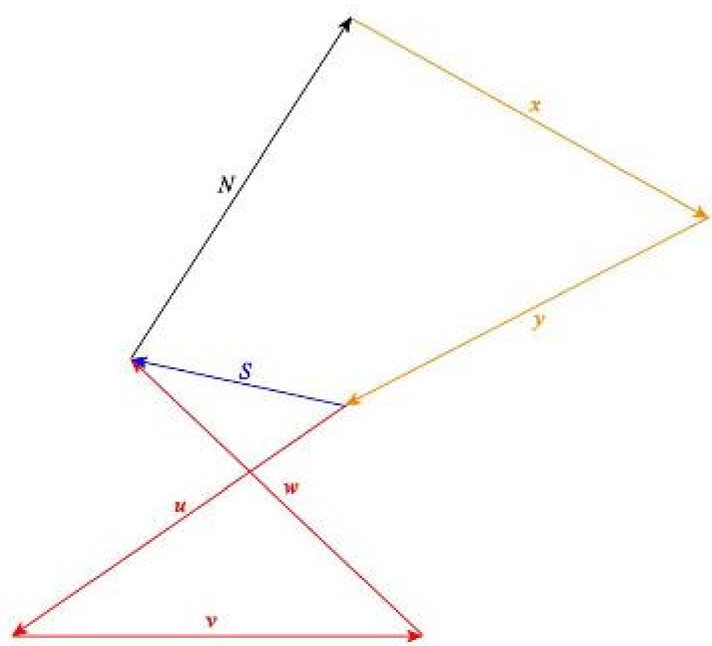
Pentagons

A degenerate pentagon inscribed in an isosceles triangle with long side to short side in the ratio of 2 : 1.

Let triangle ABC, u, v, w, P, and Q be as in the lemma and its proof. Since $PQ < 1$, we can form an isosceles triangle PQR with base PQ and equal sides of length 1. If $u + v + w = S,$ we can pick two unit vectors x and y along the equal sides of PQR, oriented so that $x + y = -S$. If S is nonzero, there is only one such pair of vectors because there are only two isosceles triangles that can complete the shape, but the sides of one of these triangles are parallel to the sides of the other. If $S = 0$, then triangle ABC is equilateral and we get a continuous family of equilateral pentagons (some of which degenerate to isosceles trapezoids). Note that when two of the vectors comprising the pentagon point in the same direction, the pentagon becomes a quadrilateral; when there are two pairs of vectors pointing in the same direction, the pentagon becomes a triangle. This latter case occurs when triangle ABC is an isosceles triangle with equal sides twice the length of the base.

Hexagons and Hexagonal Rotation

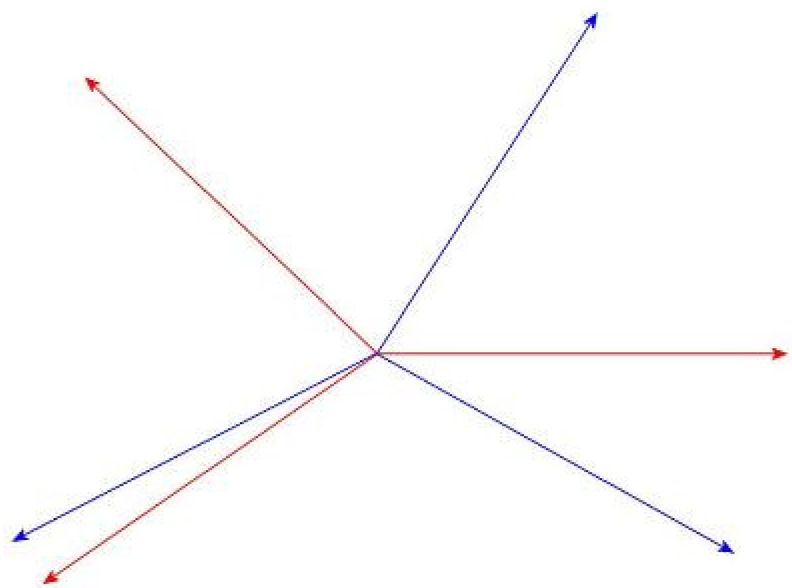
Let triangle ABC, u, v, w, P, and Q be as in the lemma and its proof. Let $S = u + v + w$. Let N be a unit vector pointing in any direction. By the triangle inequality, $|S + N| \le |S| + |N| = 2$, since we know $|S| < 1$ by the lemma. Therefore, we can form an isosceles triangle with base $S + N$ and equal sides of length 1. We can then pick two vectors x and y along the equal sides of the isosceles triangle, oriented such that $x + y = -(S + N)$, and then our hexagon can be built from the six vectors u, v, w, x, y, and N.



Vector N determines x and y because there is only one isosceles triangle with any given side lengths, and we know that two of the sides are of length 1 and the third side has length $|S + N|$. Since N can be placed in any direction, there are an infinite number of hexagons that fulfill these requirements.

Degenerate Hexagons

A degenerate hexagon is a hexagon in which two or more of the sides lie along the same line, so the hexagon is really a pentagon, quadrilateral, or triangle. In terms of vectors, a degenerate hexagon is one where two or more of the vectors *u*, *v*, *w*, *N*, *x*, and *y* point in the same direction.

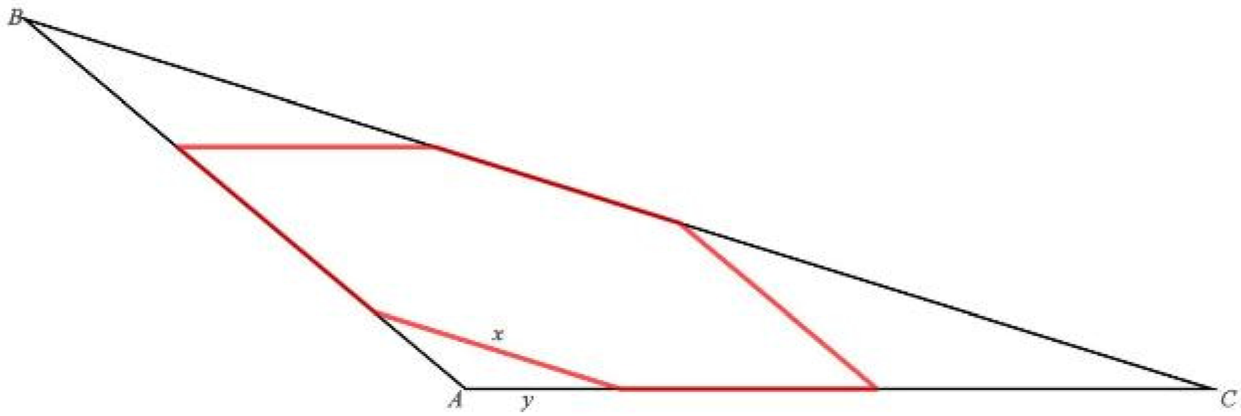


Place the three vectors *u*, *v*, and *w* (red in the figure at right) so that they start at the same point. Insert the other three vectors *N*, *x*, and *y* (blue in the figure at right) and imagine that they are free to rotate around that point. One configuration occurs when there is a blue arrow pointing in the opposite direction for each red arrow. In this configuration, there will be one blue vector in between each pair of red vectors. As we rotate the blue vectors around, they will cross over the red vectors, producing degenerate hexagons. The degenerate hexagons are triangles when all three blue vectors match the three red vectors, quadrilaterals when exactly two of the blue vectors match two of the red vectors, or pentagons when only one of the blue vectors matches a red vector. Note that the degenerate case where the hexagon is a triangle (the three blue vectors match the three red vectors) only happens in an equilateral triangle; in this case, the three red vectors must sum to zero, which means that the original triangle is equilateral.

Observe that two blue vectors can never coincide because if they did, drawing the vectors *S*, *N*, *x*, and *y* end to end would result in a triangle with a side of length 2 (or more) while its other sides (which would include *S*) would have total length less than 2, violating the triangle inequality. This implies that as the blue vectors rotate around, they all progress in the same direction – either all clockwise or all counterclockwise – though they do not generally do so with the same angular velocity. The position of one blue vector uniquely determines the other two, so two blue vectors cannot rotate in opposing directions without either violating this uniqueness or causing two blue vectors to coincide.

Hexagonal Parallelogons

Notice that every triangle contains an inscribed equilateral hexagon with parallel opposite sides. It is formed by taking *N* to be -*u*, -*v*, or -*w*. All three choices lead to the same inscribed hexagon.



Given triangle ABC, let $a = BC$, $b = AC$, and $c = AB$. Let x be the side length of the inscribed equilateral hexagonal parallelogon. Note that the hexagon cuts off three tips of the triangle, all of which are similar to the original triangle. Therefore, $a/x = b/y$, where y is as indicated in the figure above. We also have $b/(b - x - y) = c/x$. This gives us two equations in the two unknowns x and y. Solving using standard techniques, we find that $x = 1/(1/a + 1/b + 1/c)$.

Thus, for any triangle, the side length of the inscribed equilateral hexagonal parallelogon is a third of the harmonic mean of the triangle's side lengths.

Heptagons and more sides

If $n > 6$, we can use the same vector method to find the angles of the sides of an n-gon, because it is always possible to find $n - 3$ vectors whose sum is equal to a given vector of length less than 1. As in the hexagon case, there are infinitely many possible n-gons.

Conclusion

It would be interesting to see which equilateral polygons can be inscribed in polygons with more sides than a triangle, with sides of the inscribed polygons flush with the sides of the larger polygons. The method we used for triangles does not work in this more general case because our lemma does not hold for shapes other than triangles; so if we pick m unit vectors in the directions of the sides of an m-gon, their sum might have a magnitude greater than 2, in which case we would not be able to find any inscribed equilateral $(m + 2)$-gon.

Acknowledgements

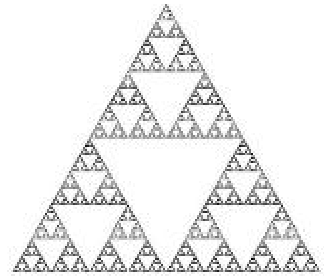
The authors thank Ken Fan and Jane Wang for their help.

Visit www.girlsangle.org/apps/InscribedEquilateralHexagons.html for an applet that enables you to see these inscribed equilateral hexagons for various triangles.

Math Buffet: Triangles!

By Ashley Wang

Three's a crowd, unless we're talking about the simplest polygon out there! Triangles not only play a major role in mathematics, but also show up in some surprising places in the real world.



The Sierpinski Triangle. See *Meditate*^{Math} on page 18.

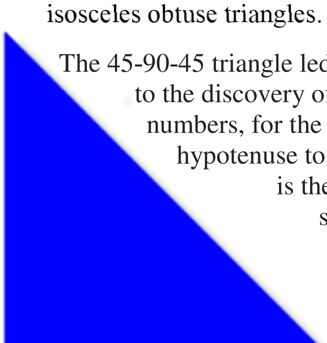
Triangles often show up in architecture. The Louvre and Egyptian pyramids feature four faces of isosceles triangles, and the Transamerica Pyramid in San Francisco is famous for its triangle-centered design. (Louvre photo courtesy of the author. Transamerica photo courtesy of C. Kenneth Fan. Giza pyramids photo by Ricardo Liberto/Wikimedia (https://creativecommons.org/licenses/by-sa/2.0/deed.en).)

Designed by I. M. Pei, the MIT Landau Building is shaped like a 30-60-90 right triangle. See *In Search of Nice Triangles* on page 22 for a special property of these triangles. (Image courtesy of Google Maps.)

One of the most beloved and unique pieces of the Boston skyline is the equilateral triangle of the Citgo sign, which itself is built from three isosceles obtuse triangles.

Geodesic spheres are actually polyhedrons made entirely of triangles. Buckminster Fuller, who lived inside a geodesic dome, popularized them in the US. (Image courtesy of C. Kenneth Fan.)

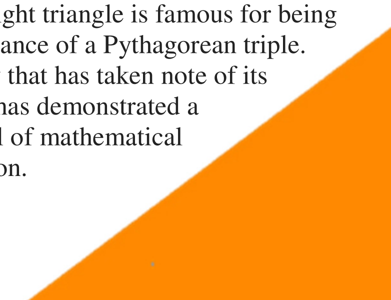
Scalene triangles can be seen all over the Charles River on warm days in the form of sails. Such sails are often right triangles.



The 45-90-45 triangle led Hippasus to the discovery of irrational numbers, for the ratio of its hypotenuse to leg length is the irrational square root of 2.

Many flags use triangles in their design (check out the flags of St. Lucia, Guyana, and Antigua and Barbuda). Shown above is the flag of the Seychelles.

The 3-4-5 right triangle is famous for being the first instance of a Pythagorean triple. Any society that has taken note of its right angle has demonstrated a certain level of mathematical sophistication.



Meditate$^{\text{Math}}$

by Girls' Angle Staff

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

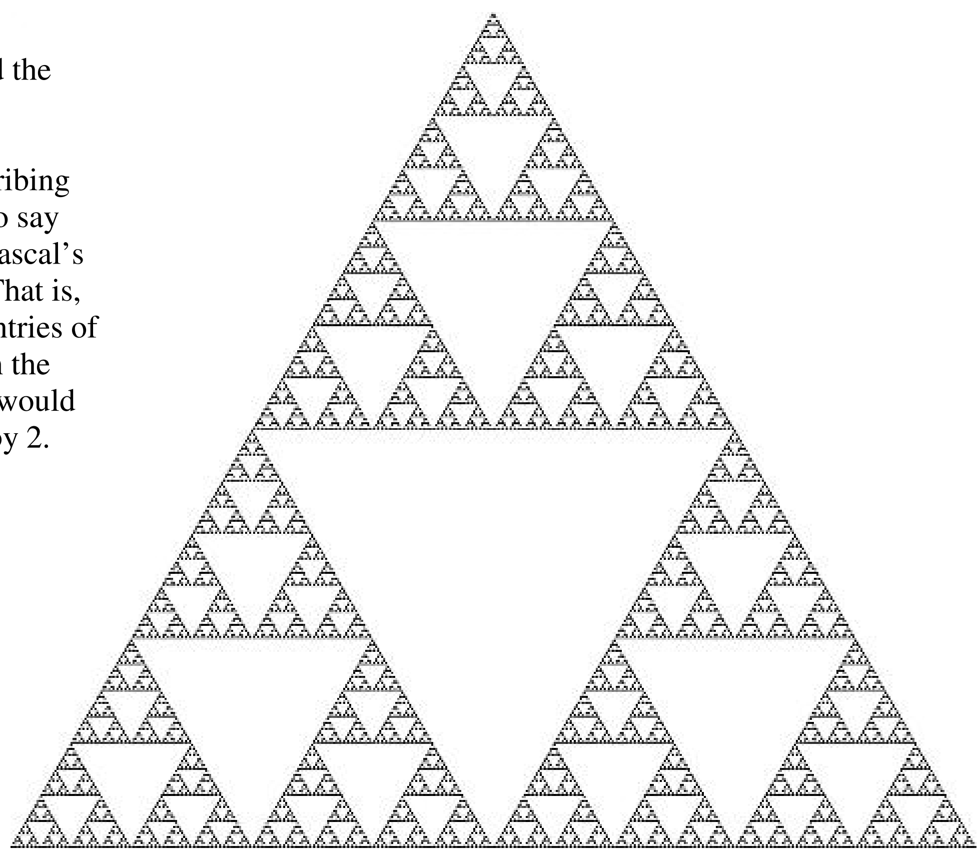
Shown at right are the first few rows of Pascal's famous triangle. Each entry is the sum of the two numbers above it. (Except for the ones, unless we imagine zeros extending to the left and right of each row.)

When you expand $(a + b)^n$, gather like terms, and list the terms so that the exponents of b increase from left to right, you will see that the coefficients correspond to rows of Pascal's triangle. For example, $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, and there is the row of Pascal's triangle that goes 1 4 6 4 1. The top row of Pascal's triangle corresponds to the expansion of $(a + b)^0$. For this reason, the rows of Pascal's triangle are numbered starting with 0 instead of 1.

1
1 1
1 0 1
1 1 1 1
1 0 0 0 1
1 1 0 0 1 1
1 0 1 0 1 0 1
1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0 1

Now let's pay attention only to the **parity** of the entries in Pascal's triangle, that is, we will only care whether an entry is even or odd. We'll rewrite Pascal's triangle putting a 1 if an entry is odd and a 0 if the entry is even.

To make what happens clearer, let's instead use a small black triangle in place of the ones and blank space for all the zeros. Drawing the first 256 rows in this manner, we get:



This triangle is called the **Sierpinski triangle**.

Another way of describing what we've done is to say that we've reduced Pascal's triangle **modulo 2**. That is, we've replaced the entries of Pascal's triangle with the remainder that entry would leave upon division by 2.

Each row now consists of a sequence of 0's and 1's.

Let's interpret these rows as binary numbers!

Doing so yields a sequence of numbers that begin like this: 1, 3, 5, 15, 17, 51, 85, 255, …. Let's call this sequence s_k, $k = 0, 1, 2, 3, \ldots$, so $s_0 = 1$, $s_1 = 3$, $s_2 = 5$, etc.

Meditate on the Sierpinski triangle and the sequence of numbers s_k. What do you see?

Here are some questions and facts to meditate upon. Can you understand and answer them?

1. All the numbers s_k are odd.

2. Every other term in the sequence s_k is a multiple of 3. More precisely, s_k is a multiple 3 if and only if k is odd.

3. The term s_k is a multiple of 5 if and only if k leaves a remainder of 2 or 3 upon division by 4.

4. A Fermat number is a number of the form $2^{2^k}+1$, where k is a nonnegative integer. The numbers s_k can be written as a product of Fermat numbers.

5. In #4, we can be quite precise about how to write s_n as a product of Fermat numbers. Let F_n be the set of Fermat numbers $2^{2^m}+1$ such that in the binary expansion of n, there is a 1 in the 2^m's place. For example, if $n = 13$, then $n = 1101$ in binary, so there is a 1 in the 2^0's place, the 2^2's place, and the 2^3's place. Therefore, $F_{13} = \{ 2^{2^0}+1,\ 2^{2^2}+1,\ 2^{2^3}+1 \}$. Then s_n is equal to the product of the Fermat numbers in F_n.

6. The Fermat numbers are, themselves, a subsequence of s_k, that is, each Fermat number can be found in the sequence s_k. For which j is $s_j = 2^{2^k}+1$?

7. Can you see that $s_0 + s_1 + s_2 + s_3 + \ldots + s_{2^n-1}$ is equal to the product

$$\prod_{k=1}^{n}(2^{2^{k-1}}+2)\ ?$$

8. In the even numbered rows (remember, the top row of Pascal's triangle is row zero), every 1 is sandwiched between two 0's (here, we imagine zeros extending to the left and right of every row).

9. How many entries in the first 2^n rows of Pascal's triangle are odd?

10. If a dart lands within the first 2^n rows of the Sierpinski triangle (with uniform probability distribution), what is the probability that the dart will hit white space?

If you have any observations of your own, we'd love to hear them!

A Self-Referential Test

by **Ghost Inthehouse**, **HolAnnHerKat**, **Katnis Everdeen**, and **Shark Inthepool**

Inspired by James Propp's self-referential multiple-choice test, four Girls' Angle club members created a self-referential test of their own, under the guidance of Girls' Angle mentor Rachel Burns, with assistance from Girls' Angle mentor Suzanne O'Meara. Can you get a perfect score on this multiple-choice test? Good luck!

---

1. The answer to Question 5 is

A) D B) A C) B D) C

---

2. The first question with the answer C is

A) 10 B) 5 C) 4 D) 3

---

3. How many answers are *not* C?

A) 6 B) 8 C) 12 D) 13

---

4. The number of answers that are B or D is

A) 7 B) 9 C) 8 D) 10

---

5. Question 1 shows that the answer to this question is

A) A B) B C) C D) D

---

6. This question has the same answer as Question 15.

A) B) C) D)

---

7. How many questions have A or D as the correct answer?

A) 9 B) 7 C) 11 D) 2

(Continued on the next page.)

8. The most common answer in this quiz is

A) C B) A C) D D) B

9. If 24 × 24 = 575, then the answer to Question 14 is not A.
What is the answer to Question 14?

A) C B) D C) A D) B

10. The answer to Question 1 is

A) A B) B C) D D) C

11. If the answer to Question 10 is A, then the answer to this question is D, and vice versa.

A) B) C) D)

12. The answer to Question 4 is

A) B B) D C) A D) C

13. The answer to this question is the number of questions whose correct answer is B.

A) 4 B) 2 C) 4 D) 3

14. The answer to Question 10 is

A) B B) D C) A D) C

15. The number of questions whose correct answer is C is

A) 1 B) 3 C) 4 D) 2

If you enjoy this kind of logic puzzle, check out James Propp's test which can be found on the internet and take a look at the self-referential true/false quiz on page 24 of Volume 10, Number 3 of this Bulletin.

In Search of Nice Triangles, Part 14

by Ken Fan | edited by Jennifer Silva

Mr. ChemCake: I am in awe of your dedication! But I'm afraid I'm going to have to kick you out soon because I have to close up for the night.

Jasmine: Emily and I are *so* close to finishing up a big project we've been working on for *so* long!

Emily: When do you have to close up?

Mr. ChemCake: In about 15 minutes. But if you don't mind working while I clean up house, you can stay until I lock up in 45 minutes or so.

Jasmine: Thanks, Mr. ChemCake! Unless we run into a big surprise, I think that should be just enough time.

Emily: I sure hope it's enough time!

Mr. ChemCake: Are you still working on that triangle problem?

Jasmine: Yes! We just completed a long computation that will hopefully enable us to find all triangles with certain nice properties. All we have to do now is translate the results of our computation back to the world of triangles.

Mr. ChemCake: I thought you might still be investigating triangles, and your determination inspired me to bake you up this little snack.

Mr. ChemCake sets down a plate of assorted cookies before Emily and Jasmine.

Emily: Thanks, Mr. ChemCake. You're so thoughtful!

Jasmine: Very thoughtful! Emily, check out the cookies!

Emily: Triangles!

Mr. ChemCake: Assorted Tirggel Triangle Treats, I call 'em.

Jasmine: They're so ornate – and delicious!

Emily: Fabulous.

Emily and Jasmine continue their investigation into nice triangles. They've been using "nice" to denote angles that measure a rational multiple of π radians.

Previously, they decided to embark on a study of the minimum polynomials of the cosines of rational multiples of π. They defined the polynomials $p_d(x)$, for $d > 1$, to be the product of all linear factors of the form $x - \cos(2\pi k/d)$, where $1 \le k \le d/2$ and $(k, d) = 1$. They defined $p_1(x) = x - 1$.

They observed that, for n odd,

$$T_n(x) - 1 = 2^{n-1} p_1(x)\left(\prod_{d|n, d>1} p_d(x)\right)^2,$$

and for n even,

$$T_n(x) - 1 = 2^{n-1} p_1(x)p_2(x)\left(\prod_{d|n, d>2} p_d(x)\right)^2,$$

where $T_n(x)$ is the nth Chebyshev polynomial of the first kind.

They showed that the minimum polynomial of $\cos(2\pi k/n)$, where k and n are relatively prime, is $p_n(x)$.

They aim to compute the constant terms of $p_n(x)$ in the hopes that doing so will enable them to determine all triangles with 3 nice angles and 2 sides of integer length.

They're also using $\Phi_n(x)$ to denote the nth cyclotomic polynomial, which is the minimum polynomial of a primitive nth root of unity.

Mr. ChemCake: I favor the obtuse ones myself. Anyway, 45 minutes whether it's Q.E.D. or not!

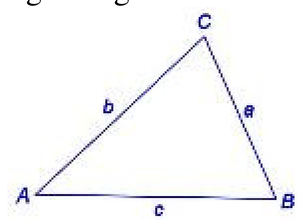
Jasmine: Got it.

Emily: Come on Jasmine, we'd better focus.

Jasmine: Okay. How do we translate these algebraic results back to triangles?

Emily: I can't remember! I need to retrace our steps to recall. We're working on the last case of our search for nice triangles: triangles with 3 nice angles and 2 sides of integer length.

Emily sketches and labels the triangle shown at right.



Emily: Let's say that a and b are the sides with integer length. Then we used the law of sines to see that the ratio of sin A to sin B must be a rational number.

Jasmine: It's coming back. Then we decided to switch perspective to complementary angles so that we could work with cosines instead of sines, because we wanted to tie this in to the minimum polynomials of the cosines of nice angles. That is, if we let $X = \pi/2 - A$ and $Y = \pi/2 - B$, then we must find pairs of nice angles X and Y such that the ratio of cos X to cos Y is rational.

Emily: Which we may now be able to do! We know that if $p(x)$ is the minimum polynomial of cos X and $q(x)$ is the minimum polynomial of cos Y, and cos $X = r$ cos Y, where r is a nonzero rational number, then

$$q(x) = p(rx)/r^d,$$

where d is the degree of $p(x)$. By examining the constant terms of the minimum polynomials of cosines of nice angles, we now know that there are exactly six cases where $p_n(x) = p_m(rx)/r^{\varphi(m)/2}$, where $n \neq m$ and the polynomial $p_k(x)$ is the minimum polynomial of $\cos(2\pi/k)$. They are

n	***m***	***r***
1	2	-1
1	3	-1/2
1	6	1/2
2	3	1/2
2	6	-1/2
3	6	-1

Jasmine: For completeness, we mustn't neglect the case where $n = m$.

Emily: I agree. So let's do that $n = m$ case now. When $n = m$, $r = -1$ or 1, so we either have cos X = cos Y or cos X = -cos Y.

Jasmine: If cos X = cos Y, then either X and Y differ by a multiple of 2π radians, or X and $-Y$ differ by a multiple of 2π radians. That is, either $X = Y + 2\pi k$ or $X = -Y + 2\pi k$, where k is an integer. If A and B are the complements of X and Y, respectively, then this means that either

$$\pi/2 - A = \pi/2 - B + 2\pi k \text{ or } \pi/2 - A = B - \pi/2 + 2\pi k,$$

which are equivalent to either

$$A = B + 2\pi k \text{ or } A = \pi - B + 2\pi k,$$

for some integer k. [The k's in these statements are not the same, but they all stand for integers.]

Emily: But A and B are angles in a triangle so both are between 0 and π radians. In fact, their sum must be less than π radians since the sum of the angles in a triangle is π radians. So in the case where A and B differ by a multiple of 2π radians, we must, in fact, have $A = B$; this corresponds to the isosceles case, which we already knew about.

Jasmine: And if A and $\pi - B$ differ by a multiple of 2π radians, then we must have $A = \pi - B$, which isn't possible since $A + B < \pi$.

Emily: And if $\cos X = -\cos Y$, then either X and $\pi - Y$ differ by a multiple of 2π radians, or X and $Y - \pi$ differ by a multiple of 2π radians. If we again denote the complements of X and Y by A and B, respectively, this means that either A and $-B$ differ by a multiple of 2π radians, or that A and $B + \pi$ differ by a multiple of 2π radians.

Jasmine: If $A = -B + 2\pi k$ for some integer k, then $A + B = 2\pi k$; but this is impossible since A and B are angles in a triangle, so must satisfy $0 < A + B < \pi$. And if $A = B + \pi + 2\pi k$ for some integer k, then we cannot have both A and B be between 0 and π; for if they were, then $-\pi < A - B < \pi$.

Emily: So the only nice triangles we get from the case where $n = m$ are the nice isosceles triangles. Such a triangle can have any nice angle between 0° and 180° as its apex angle, and we can scale the triangle as necessary so that its two equal sides have any integer length we please.

Jasmine: Good, that takes care of the case $n = m$.

Emily: Let's turn to the six cases where n is not equal to m.

Jasmine: The first case in our table is $n = 1$, $m = 2$, and $r = -1$. The polynomial $p_1(x) = x - 1$ is the minimum polynomial of cos 0, so $X = 2\pi k$, for some integer k.

Emily: And the polynomial $p_2(x) = x + 1$ is the minimum polynomial of $\cos \pi$, so $Y = \pi + 2\pi j$, for some integer j. But no complement of an angle of the form $\pi + 2\pi j$ can live inside a triangle, so this case doesn't yield a nice triangle.

Emily: In fact, that observation eliminates all the cases where n or m is 2, so we really only have three cases left to check, namely $(n, m, r) = (1, 3, -1/2), (1, 6, 1/2)$, or $(3, 6, -1)$.

Jasmine: In the case where $n = 1$, $m = 3$, and $r = -1/2$, we have $X = 2\pi k$ and $Y = \pm 2\pi/3 + 2\pi j$, where k and j are integers. The only complement of an angle of the form $2\pi k$ that can live in a triangle is $\pi/2$ – the right angle.

Emily: And the only complement of an angle of the form $\pm 2\pi/3 + 2\pi j$ that can live in a triangle is … actually, there isn't one! After all, $\pi/2 - 2\pi/3 + 2\pi j = -\pi/6 + 2\pi j$, which is never between 0 and π, and $\pi/2 + 2\pi/3 + 2\pi j = 7\pi/6 + 2\pi j$, which is also never between 0 and π. So we can eliminate the cases where n or m is equal to 3 as well!

Jasmine: Gosh! That leaves us with just one case left to consider: $n = 1$, $m = 6$, and $r = 1/2$.

Emily: In that case, we again have A equal to the right angle, but now $Y = \pm\pi/3 + 2\pi j$, where j is an integer.

Jasmine: The only complements of $\pm\pi/3 + 2\pi j$ that can live inside a triangle are $\pi/6$ and $5\pi/6$ radians, or 30° and 150°.

Emily: Since A is already 90°, we can't have B = 150° since the sum of A and B must be less than 180°.

Jasmine: So aside from the isosceles cases, the only triangle that can have 3 nice angles and 2 sides of integer length is, indeed, the 30-60-90 right triangle! You've suspected that for quite some time. You were correct!

Emily: Well, it was less suspicion than just wondering. But now we know!

Jasmine: Yes, now we *know*! We did it! We did it!

Mr. ChemCake: What's the commotion?

Emily: We solved our problem! Our search for nice triangles is complete!

Mr. ChemCake: I had a feeling that's what the excitement was about, so I made a classic hot fudge sundae for each of you to celebrate. Congratulations!

Emily: Thanks, Mr. ChemCake. You're so kind to us!

Mr. ChemCake: Well, I'm not that kind, 'cause I'm kicking you out! You're going to have to take those to go.

Jasmine: Oh goodness, right! Sorry about that!

Emily: Hey, there's our bus! Hurry, Jasmine!

Emily and Jasmine gather up their papers, bid adieux to Mr. ChemCake, and head out with their to-go sundaes. The next day, they write the summary of their results on nice triangles, which appears on the next page.

A Classification of Nice Triangles

by Emily and Jasmine

Number of nice angles	Number of integer sides	Possibilities
3	3	Equilateral triangles: All angles are 60° and the 3 equal sides can have length of any positive integer.
3	2	Isosceles triangles: Any nice angle can serve as apex angle and the equal sides can have any positive integer length. The 30-60-90 triangle: The short leg can have length any positive integer and the hypotenuse will have length twice that of the short leg.
3	1	Any triangle with nice angles can be scaled to make any one of its sides have length of any given integer.
2	1, 2, or 3	A triangle with 2 nice angles has 3 nice angles, so these cases revert to those above.
1	3	Triangles with a 60° angle: Let p and q be relatively prime integers with $0 < p < q$. If $p = q$ (mod 3), then $(p^2 + 2pq)/3$, $(2pq + q^2)/3$, and $(p^2 + pq + q^2)/3$ can be taken as the sides of a primitive such triangle. Otherwise, $p^2 + 2pq$, $2pq + q^2$, and $p^2 + pq + q^2$ are the sides of a primitive such triangle. All primitive such triangles can be obtained in this way. Triangles with a right angle (Pythagorean triples)[1]: Let p and q be relatively prime integers with $0 < p < q$ and with p and q not both odd. Then $q^2 - p^2$, $2qp$, and $q^2 + p^2$ can be taken as the sides of a primitive such triangle. All primitive Pythagorean triples can be obtained in this way. Triangles with a 120° angle: Every such triangle can be obtained by removing an equilateral triangle from a nice triangle with a 60° angle and integer side lengths.
1	1 or 2	For any nice angle A and positive integers b and c, one can form a triangle with sides of length b and c meeting at an angle with measure A.
0	3	Any triples of integers that satisfy the triangle inequalities can be used as the side lengths of a triangle with integer side lengths. The number of different such triangles with perimeter p is given by the following chart:

p mod 12	**No. of Δ's**	**p mod 12**	**No. of Δ's**
0	$p^2/48$	6	$(p^2 + 12)/48$
1	$(p^2 + 6p - 7)/48$	7	$(p^2 + 6p + 5)/48$
2	$(p^2 - 4)/48$	8	$(p^2 - 16)/48$
3	$(p^2 + 6p + 21)/48$	9	$(p^2 + 6p + 9)/48$
4	$(p^2 - 16)/48$	10	$(p^2 - 4)/48$
5	$(p^2 + 6p - 7)/48$	11	$(p^2 + 6p + 5)/48$

[1] Pythagorean triples are well studied, and both Emily and Jasmine had seen treatments of the subject before so they didn't address them in detail during their search for nice triangles. The formulas given here are well known and included for the sake of completeness.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 21 - Meet 8
November 2, 2017
Mentors: Karia Dibert, Neslly Estrada, Danielle Fang, Molly Humphreys, Suzanne O'Meara, Kate Pearce, Jacqueline Shen, Jane Wang, Josephine Yu

Polygons, Platonic solids, induction, invariants, Fibonacci numbers, pizza numbers, perspective geometry, coin flipping problems, self-referential tests, greatest common factors, and an epic round of number hot potato filled up this meet.

Session 21 - Meet 9
November 9, 2017
Mentors: Rachel Burns, Sarah Coleman, Karia Dibert, Neslly Estrada, Danielle Fang, Molly Humphreys, Suzanne O'Meara, Kate Pearce, Jane Wang

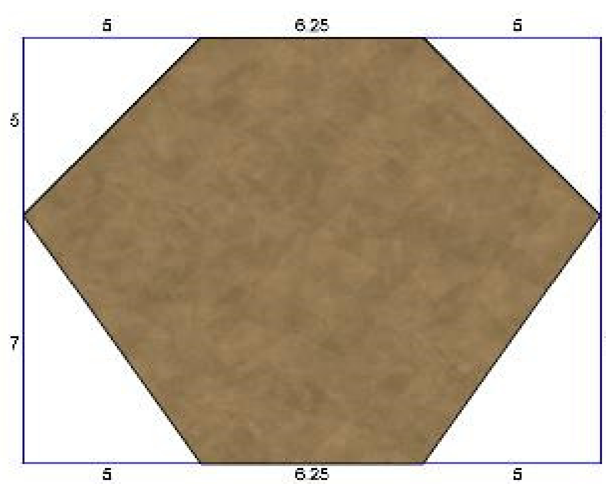
Part of what mathematics is about is learning to deduce general features from specific examples. When we understand, for instance, what is true for all numbers instead of for a specific number, we gain in perspective. Some of our members are beginning to write proofs, and for some of them, the procedure has been to verify specific instances of a theorem and then try to generalize the specific computations to create a general proof. This process requires understanding when an operation applies generally or only to a specific case. By practicing, one becomes increasingly aware of when one is using a general principle or not. The notion of a variable greatly assists this process because it encourages one to think of numbers in general terms. Variables also help us to keep our thoughts organized by enabling us to keep track of important quantities.

Four members finished creating their own self-referential test. See page 20.

Session 21 - Meet 10
November 16, 2017
Mentors: Rachel Burns, Sarah Coleman, Karia Dibert, Danielle Fang, Alexandra Fehnel, Suzanne O'Meara, Kate Pearce, Jacqueline Shen, Sarah Tammen, Josephine Yu

For this meet, **Lucy Kona**'s mom baked us a scrumptious brownie. But it wasn't a rectangular brownie. Instead, it was an irregular hexagon, as shown at right. Unlike a regular hexagon, this brownie's sides and angles were not all equal, but it did retain an axis of mirror symmetry. The measurements are given in inches. Nobody could eat any brownie until we had a way to split this brownie into 36 equal pieces. Several members tackled that problem.

Can you figure out a way?



Session 21 - Meet 11   Mentors: Rachel Burns, Neslly Estrada, Alexandra Fehnel,
November 30, 2017   Molly Humphreys, Suzanne O'Meara, Kate Pearce,
Jacqueline Shen, Jane Wang, Josephine Yu

One of our members is hot on the trail of figuring out how to construct a regular pentagon with compass and straightedge.

Session 21 - Meet 12   Mentors: Sarah Coleman, Karia Dibert, Anna Ellison,
December 7, 2017   Neslly Estrada, Alexandra Fehnel, Suzanne O'Meara,
Kate Pearce, Ashley Wang, Jane Wang

For the first time at Girls' Angle, our traditional end-of-session Math Collaboration was created by one of our mentors. Alexandra Fehnel dreamed up a spectacular one! Touching on all mathematics covered this fall, Alexandra presented the girls with 60 math problems and 10 mysterious polyominoes. Each polyomino turned out to be the net of a cube. The outsides of these cubes sported numbers on each face, like a standard die, only the numbers were not the numbers 1 through 6. An interior face of the cube was decorated with a 5 by 5 grid of numbers as if the cube were equipped with an internal control panel.

With no instruction, the girls had to figure out how making sense of all this would enable them to break into a treasure chest. And, after a number of Aha! moments, they succeeded!

Here's a sampling of the problems from Alexandra's amazing activity:

Find the missing whole number: 118,587,876,497 = $17^?$.

The cyclotomic polynomial whose roots consists of the primitive nth roots of unity is

$$x^{54} + x^{27} + 1.$$

What is n?

In the identity $\sqrt{44g^7w^3} = 2g^2w\sqrt{Xg^Yw^Z}$, where g and w are variables, what is XYZ?

Solve for x: $\log_5 x = 2$.

A perspective drawing of a scene is meant to be viewed from a distance of 6 feet from the canvas. In the canvas, a 6 foot tall person is depicted standing upright. When the drawing is viewed from the ideal distance of 6 feet, the depicted person appears to be standing 18 feet behind the canvas. How tall is the drawing of the person in inches?

Complete the row of Pascal's triangle that begins 1, 6, ….

What is the maximum number of pieces you can cut a pizza into using 7 straight cuts?

What is the next number after 1 which is both a perfect square and a triangular number?

Calendar

Session 21: (all dates in 2017)

September	7	Start of the twenty-first session!
	14	
	21	No meet
	28	
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	23	Thanksgiving - No meet
	30	
December	7	

Session 22: (all dates in 2018)

February	1	Start of the twenty-second session!
	8	
	14	
	22	No meet
March	1	
	8	
	15	
	22	
	29	No meet
April	5	
	12	
	19	No meet
	26	
May	3	
	10	

Tenth grade girls! You can participate in SUMIT 2018 on either February 3 or 4, 2018. Register now! It's a mathematical adventure you won't forget. For more info, please visit www.girlsangle.org/page/SUMIT/SUMIT.html.

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

America's Greatest Math Game: Who Wants to Be a Mathematician.

(advertisement)

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

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The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is located about 12 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, lecturer, Harvard University
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, Instructional Designer, Stanford University
Lauren McGough, graduate student in physics, Princeton Univeresity
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, professor of aeronautics and astronautics, MIT
Lauren Williams, professor of mathematics, UC Berkeley

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

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Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets)	☐ I will pay on a per meet basis at $30/meet.
☐ I will pay on a per meet basis at $20/meet.	☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________