Girls' Angle Bulletin

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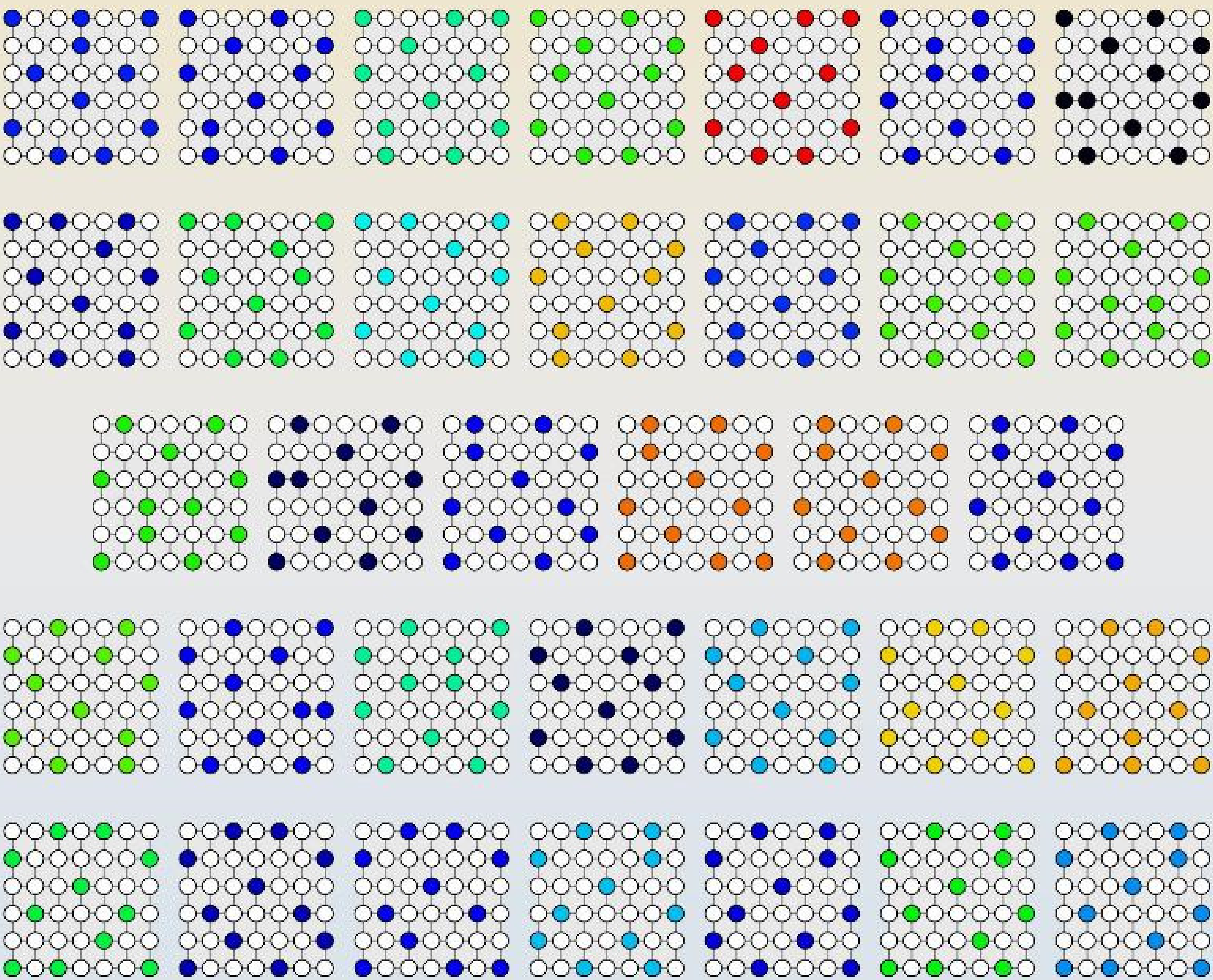
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

Doing math, a lot of the time it feels like nothing is working. There's no progress, no new observations. So it's good to develop ways to handle that feeling so that you do not give up, such as thinking about other problems, reading related math, taking walks, etc. -Ken Fan, President and Founder

Girls' Angle Bulletin

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Email: girlsangle@gmail.com

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Editors: Amanda Galtman
Jennifer Sidney
Executive Editor: C. Kenneth Fan

Girls' Angle:
A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: *Complete Domination* by C. Kenneth Fan. For more on dominating sets, see *Optimal Resource Placement: From Disneyland to Dominating Sets, Part 2* on page 8.

An Interview with Karen Lange, Part 4

If you want to have fun being successful in math, I think it's incredibly helpful to cultivate your community of people you talk math with, because persisting alone is very difficult.

This is the concluding part of our four-part interview with Prof. Karen Lange of Wellesley College.

Ken: It sounds like there might even be applications of computability theory to number theory, because they're dealing with subsets of natural numbers. For example, what natural numbers are representable by certain binary forms?

Karen: Yes, there are whole research subfields that ask: what are computable real numbers? People are very interested in these kinds of things.

Ken: What specifically are you interested in?

Karen: Right now, a lot of what I do is search for cool mathematical problems. We talked about how, although every vector space has a basis, when you try to translate that to the computable world, the theorem falls apart. It's true existentially, but not computably. So, a lot of what I do is look for interesting mathematical problems and try to calibrate just how hard they are.

I can tell you a structure that I really like, that I want to know the answers about, but I don't. This open question has been open for me for way too long, and I'm frustrated about it, so if people get into it, they should tell me what they learn.

Since computability is an area of logic, I often draw from structures and problems that are interesting to other people in other areas of logic. We all love the real numbers. Reals are a crucially important field.

Think about the properties of the reals. It's an ordered field. It's Archimedean in the sense that if I add one positive element to itself, I can get above any other positive element. Earlier, you mentioned the algebraically closed fields. Well, you could also think about real closure, instead. Because if you want to keep an ordered field, you're not going to be able to algebraically close it and preserve an ordering, because you'll be in a complex number situation.

Using as your model example the real numbers, people came up with the idea of real closed fields. They're ordered fields that are closed under taking roots of odd polynomials — every odd polynomial has a root — and you have square roots for positive numbers. You can't take the square root of a negative number in the reals, but you can take the square root of 2 or any positive number.

So, people study real closed fields, just like they study algebraically closed fields. And one structure, or a substructure of real closed fields that I became very interested in, is something called "integer part." As we know, the integers sit inside the reals, and the integers are like a backbone or skeleton for the real numbers.

What properties do integers have? Well, they have addition and multiplication. They're a ring, not a field. They're ordered, because they're sitting inside this ordered field. Also, every real number is within distance one of an integer. That's why integers are kind of a skeleton for the reals.

You could generalize and take any real closed field. It's something that looks a lot like the reals. It's got an ordering on it.

You've got all these roots. It's not necessarily complete in the analysis sense, because that gets you into uncountable territory. But then, as I mentioned, the reals are Archimedean. You have this nice property where, from any positive real number, if you add that number to itself enough times, you can get above any fixed positive number.

In general, real closed fields don't necessarily have this property. There are elements with qualitatively different magnitudes. If you add 1 to itself and get all the natural numbers, you might not get above all elements in this real closed field.

So there are infinite elements and infinitesimal elements running around in some of these real closed fields. Once you have infinite elements, you could ask yourself: Is there a backbone, like the integers were for the real numbers? It won't just be the integers anymore, because there are elements that are bigger than all whole numbers.

And, in fact, there is a ring, a discrete-ordered ring, where every element of the real closed field is within distance one of some element of the ring. A lovely proof was produced by Mourgues and Ressayre in the '90s.

It's an amazing result, because when you start to think about building the integer part, some naïve ways of building it fail. So, there is an existence proof and there's somewhat of a construction, but it's a very complicated construction. Moving to the computable context, my collaborators and I proved that there are computable real closed fields that don't have computable integer parts.

What we don't know is, how hard computationally is it to find an integer part of a given computable real closed field? We know that you're going to need at least the power of the halting problem, but we don't know the exact complexity of the problem.

Ken: Are you saying that if you somehow had a device that could answer the halting question, then suddenly, you would be able to compute more of these integer parts, but it still doesn't get you all cases, and you're not sure what you need to have so that you can say, "Yes," to all computable real closed fields, with these tools, one can compute an integer part for it.

Karen: This is on the right track of reasoning, but let me make it more clear. What we don't know is if the halting problem is sufficient. It might be that one could always use the halting problem to compute one of these integer parts. But we don't know yet.

Think of it as a lower-bound/upper-bound issue. What's the complexity of integer parts? I know that there are some real closed fields out there, where I'm going to need at least the halting problem, so anything below the halting problem isn't going to cut it.

As for an upper bound, what we can guarantee is sufficient is some ridiculously high upper bound. We should be able to get these bounds closer to each other, but the truth is we haven't yet.

Ken: You point out this hierarchy of complexity. It's hard for me to imagine what is less complex than the halting problem, but still allows you to compute some of the non-computable sets?

Karen: Actually, this field is very young, but one of the big early questions was: Can you get a subset of natural numbers that is strictly of lower complexity than the halting set? And you can. In fact, the different levels of complexity in between is dense.

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Thank you and best wishes,
Ken Fan
President and Founder
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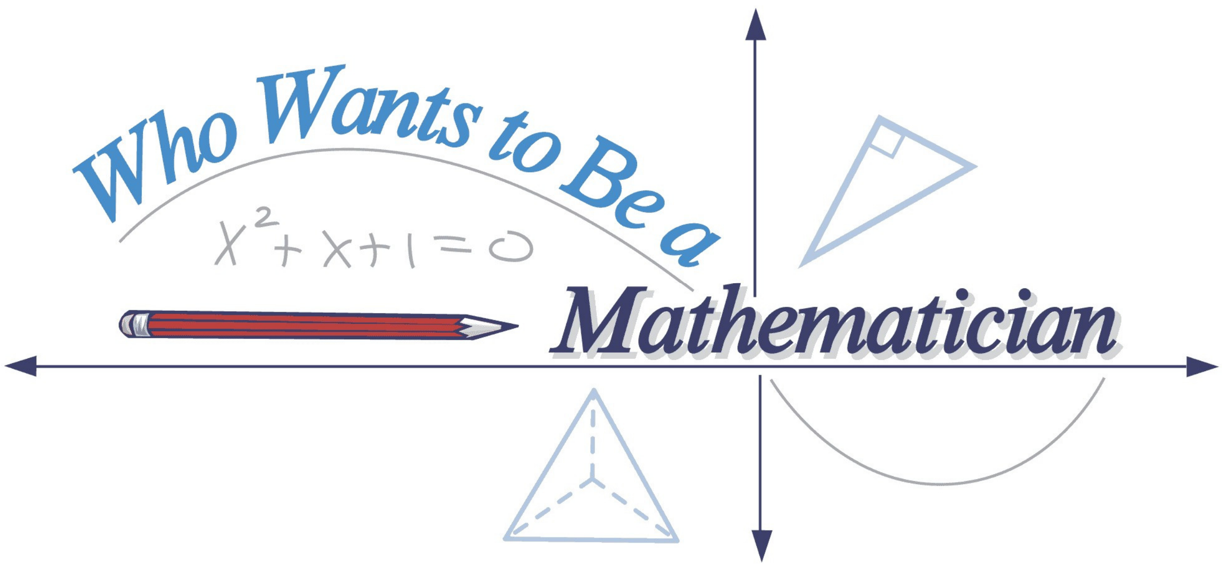

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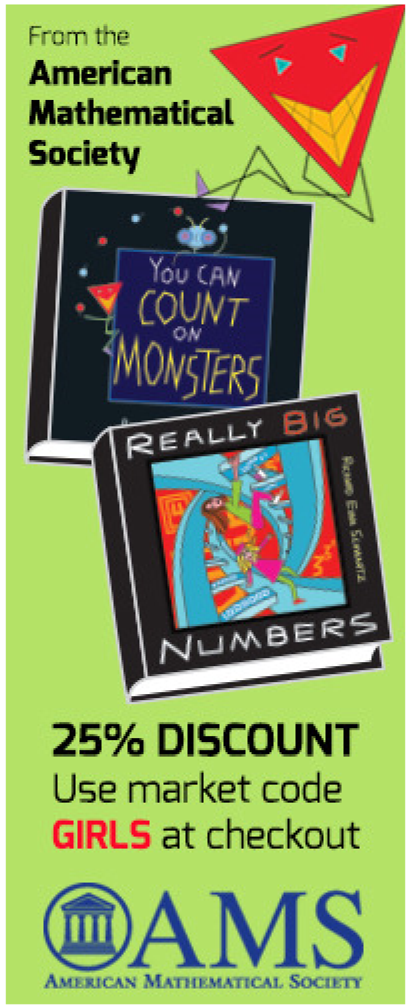

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Optimal Resource Placement:
From Disneyland to Dominating Sets, Part 2

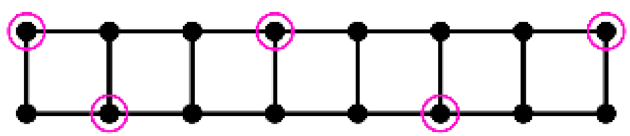
by Jillian Cervantes and Pamela E. Harris[1][2]

In this installment, we'll examine in more detail the theorem by Jacobson and Kinch we mentioned at the end of Part 1, restated below [2]:

Theorem. We have $\gamma(2 \times n) = \left\lceil \frac{n+1}{2} \right\rceil$.

(For the definition of the graph $2 \times n$, see the start of the proof below.) Here, $\lceil x \rceil$ denotes the ceiling function, i.e., the smallest integer greater than or equal to x.

The activity about cell phone tower placement at the end of Part 1 is answered by Jacobson and Kinch's theorem, which tells us that for $n = 8$, we have $\gamma(2 \times 8) = 5$. Did you have that number of cell phone towers? Yay! In fact, a placement of the 5 towers is illustrated below.

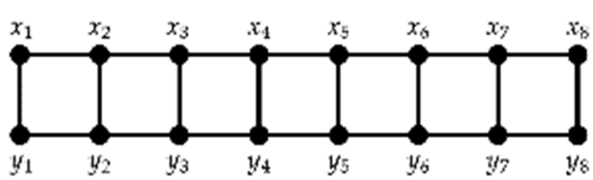


How did they figure that formula out? How can we know with mathematical certainty that 5 is the optimal number of vertices?

Note that proving the equality in the theorem above is equivalent to proving that $\lceil (n+1)/2 \rceil$ is both an upper bound and a lower bound for $\gamma(2 \times n)$.

We first prove it is a lower bound, that is, $\gamma(2 \times n) \le \lceil (n+1)/2 \rceil$. We consider cases depending on whether n is even or odd.

Proof. Consider the graph $2 \times n$ with vertices labeled $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ with edges connecting x_i, y_i ($1 \le i \le n$), edges connecting consecutive vertices x_i and x_{i+1} ($1 \le i \le n - 1$), as well as edges connecting consecutive vertices y_i and y_{i+1} ($1 \le i \le n - 1$).

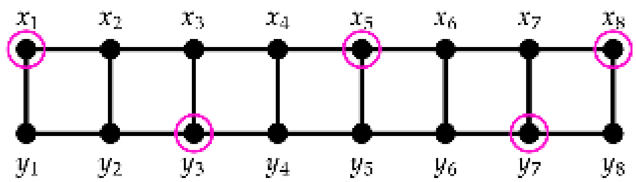


[1] Both authors are from the Department of Mathematical Sciences at the University of Wisconsin Milwaukee.
[2] This publication supported in part by a grant from MathWorks.

Case 1: Suppose n is even. Let S consist of the vertices x_i, y_j such that $i = 1 \pmod 4$ and $j = 3 \pmod 4$, together with vertex x_n.

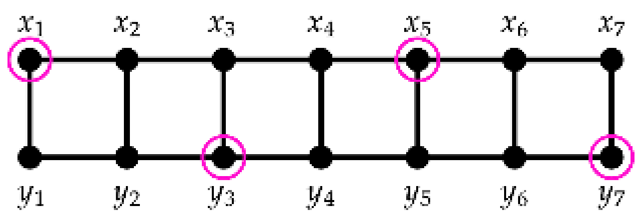
(The notation $a = b \pmod m$ means that m divides evenly into $b - a$ and is read "a equals b modulo m." For more on modular arithmetic we recommend [3].)

For our 2×8 grid graph, S consists of the circled vertices in the following graph:



Please verify that S contains $\lceil (n+1)/2 \rceil$ vertices and dominates the $2 \times n$ grid graph. This shows that for n even, $\gamma(2 \times n) \le \lceil (n+1)/2 \rceil$.

Case 2: Suppose n is odd. Let S consist of the vertices x_i, y_j such that $i = 1 \pmod 4$ and $j = 3 \pmod 4$.



Again, please verify that S dominates $2 \times n$ and S contains $\lceil (n+1)/2 \rceil$ vertices. Thus, when n is odd, we also have $\gamma(2 \times n) \le \lceil (n+1)/2 \rceil$. □

To complete the proof of Jacobson and Kinch's theorem, we would next need to show that $\gamma(2 \times n) \ge \lceil (n+1)/2 \rceil$. Rather than proving this inequality for the general n, we will prove the special case of $\gamma(2 \times 8) \ge 5$. We leave the general proof for you to explore and complete.

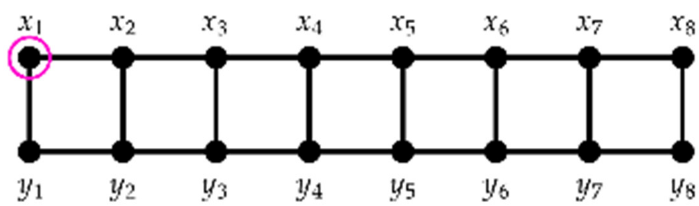
Remark. Rigorously proving a lower bound for $\gamma(G)$ is much more difficult than proving an upper bound. This is because for any proposed lower bound L, we must check that no subset of vertices of size $\ell < L$ dominates the graph. Note that this may involve testing $\binom{|V(G)|}{L-1}$ subsets, where $|V(G)|$ is the number of vertices in G. For example, for the 2×8 grid graph with 16 vertices, we would need to test $\binom{16}{4}$ = 1,820 subsets to check that any placement of 4 towers on the graph do not dominate! This is why computer-aided proofs can be so useful for proving lower bounds. In the following proof, however, we give an argument that allows us to avoid exhaustively checking every subset of 4 vertices.

Proceeding with the proof of the 2×8 case, we must show that there is no way we can arrange 4 vertices to dominate the 2×8 graph.

Proof. Let's consider dominating the 2×8 graph with 4 vertices. We consider two cases: Either we select a corner vertex or we do not.

Case 1: Select a corner vertex.

Because of the symmetry of the graph, we may, without loss of generality, assume x_1 is selected.

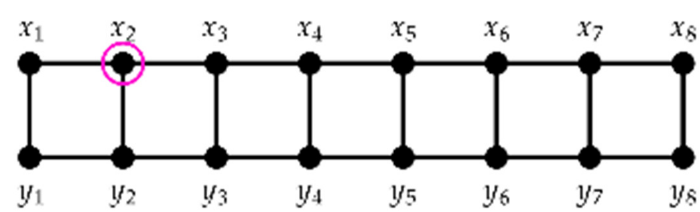


Note that x_1 dominates 3 vertices, namely, x_1, x_2, and y_1. Then the 13 remaining vertices must be dominated by selecting 3 vertices.

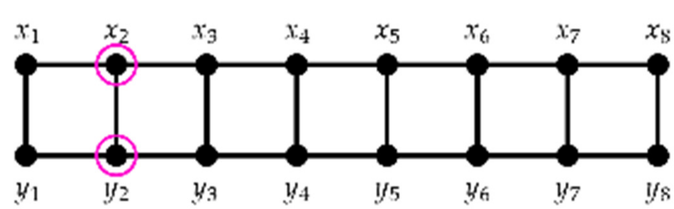
However, in a $2 \times n$ graph, any selected vertex dominates at most 4 vertices (3 if it is a corner vertex, otherwise, it dominates itself, the vertex vertically across from it, and the vertices directly to the right and left). Thus, 3 vertices can dominate at most $3 \times 4 = 12$ vertices. Therefore, the graph cannot be dominated using 4 vertices in this case.

Case 2: Do not select a corner vertex.

We work left to right, dominating the leftmost vertices first. Notice that for x_1 to be dominated, we must select x_2, since in this case selecting the corner vertex y_1 is not an option.



Now, y_1 is still not dominated. Our only option to dominate y_1 without selecting a corner vertex is to select y_2.



Selecting x_2 and y_2 dominates 6 vertices: x_1, x_2, y_1, y_2, x_3, and y_3. There are 10 vertices that are yet to be dominated. We now determine if there is any way to configure the remaining 2 vertices to dominate those vertices of the graph.

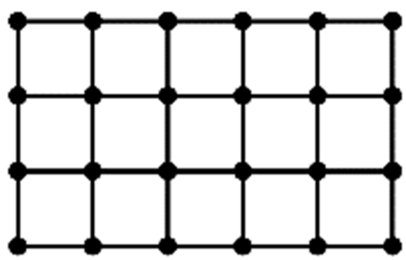
Notice that any selected vertex will dominate 4 vertices, since we are not selecting corner vertices. Then any 2 vertices we select will dominate $2 \times 4 = 8$ vertices. Therefore, this graph cannot be dominated with 4 vertices, and so $\gamma(2 \times 8) \geq 5$. ☐

Although we proved in general that $\gamma(2 \times n) \leq \lceil (n+1)/2 \rceil$, we have not proved in general that $\gamma(2 \times n) \geq \lceil (n+1)/2 \rceil$. How would you go about proving the lower bound for the general $2 \times n$ graph?

Try to complete the proof and conclude that $\gamma(2 \times n) = \lceil (n+1)/2 \rceil$.

In the time since Jacobson and Kinch's 1983 paper, domination numbers have also been established for larger grid graphs. David C. Fisher gives a table of domination numbers for grid graphs of size up to 21×21, which we reproduce on the next page.

Challenge problem: According to Fisher's result, $\gamma(4 \times 6) = 7$. Can you find an arrangement of 7 towers which dominate the graph below?



Next time, we will give you the answer as well as explore a variant of domination called (t, r) broadcast domination.

References

[1] Gonçalves, Daniel, et al. "The domination number of grids." *SIAM Journal of Discrete Mathematics*, vol. 25, no. 3, 2011, pp. 1443-1453. *arxiv*.

[2] Jacobson, Michael, and Lael Kinch. "On the domination number of products of a graph; I." *Ars Combinatoria*, vol. 18, 1984, pp. 33-44. *Science Direct*.

[3] Markan, Sean. "A Modular Arithmetic Primer." *Sean Markan's Homepage*, 2021, markan.net/mods.html. Accessed 28 December 2023.

Theorem. *For all $n \le m$ and $n \le 21$, we have:*

$$\gamma(G_{n,m}) = \begin{cases} \left\lceil \frac{m}{3} \right\rceil & \text{if } n = 1 \\ \left\lceil \frac{m+1}{2} \right\rceil & \text{if } n = 2 \\ \left\lceil \frac{3m+1}{4} \right\rceil & \text{if } n = 3 \\ m+1 & \text{if } n = 4 \text{ and } m = 5,6,9 \\ m & \text{if } n = 4 \text{ and } m \ne 5,6,9 \\ \left\lceil \frac{6m+4}{5} \right\rceil - 1 & \text{if } n = 5 \text{ and } m = 7 \\ \left\lceil \frac{6m+4}{5} \right\rceil & \text{if } n = 5 \text{ and } m \ne 7 \\ \left\lceil \frac{10m+4}{7} \right\rceil & \text{if } n = 6 \\ \left\lceil \frac{5m+1}{3} \right\rceil & \text{if } n = 7 \\ \left\lceil \frac{15m+7}{8} \right\rceil & \text{if } n = 8 \\ \left\lceil \frac{23m+10}{11} \right\rceil & \text{if } n = 9 \\ \left\lceil \frac{30m+15}{13} \right\rceil - 1 & \text{if } n = 10 \text{ and } m =_{13} 10 \text{ or } m = 13,16 \\ \left\lceil \frac{30m+15}{13} \right\rceil & \text{if } n = 10 \text{ and } m \ne_{13} 10 \text{ and } m \ne 13,16 \\ \left\lceil \frac{38m+22}{15} \right\rceil - 1 & \text{if } n = 11 \text{ and } m = 11,18,20,22,33 \\ \left\lceil \frac{38m+22}{15} \right\rceil & \text{if } n = 11 \text{ and } m \ne 11,18,20,22,33 \\ \left\lceil \frac{80m+38}{29} \right\rceil & \text{if } n = 12 \\ \left\lceil \frac{98m+54}{33} \right\rceil - 1 & \text{if } n = 13 \text{ and } m =_{33} 13,16,18,19 \\ \left\lceil \frac{98m+54}{33} \right\rceil & \text{if } n \ne 13 \text{ and } m \ne_{33} 13,16,18,19 \\ \left\lceil \frac{35m+20}{11} \right\rceil - 1 & \text{if } n = 14 \text{ and } m =_{22} 7 \\ \left\lceil \frac{35m+20}{11} \right\rceil & \text{if } n = 14 \text{ and } m \ne_{22} 7 \\ \left\lceil \frac{44m+28}{13} \right\rceil - 1 & \text{if } n = 15 \text{ and } m =_{26} 5 \\ \left\lceil \frac{44m+28}{13} \right\rceil & \text{if } n = 15 \text{ and } m \ne_{26} 5 \\ \left\lceil \frac{(n+2)(m+2)}{5} \right\rceil - 4 & \text{if } n \ge 16 \end{cases}$$

Interesting historical note: Many people cite this in a manuscript by Fisher, but upon searching, we could not locate the manuscript. Instead we cite [1], which identifies this result as one in a manuscript of Fisher.

Permutations and Basic Group Theory: Part 1[1]

by Robert Donley[2]
edited by Amanda Galtman

While recent installments focused on compositions and partitions, these structures now provide a motivated introduction to our next main topic, permutations. As originally introduced in "Shortcuts to Counting" (see Volume 15, Number 3 of the Girls' Angle Bulletin), a permutation is an ordered list of distinct objects. In what follows, we attempt to quantify basic properties of permutations, with an eye towards studying all permutations together. These ideas lead to the beginnings of **group theory**, the mathematical theory of symmetries.

We maintain the definitions and notation for partitions and compositions from previous installments. Every partition of k is also a composition of k with the same number of parts, and, given such a composition, we obtain a unique partition by ordering its parts. Recall that the number of weak partitions of k with n parts is the binomial coefficient $\binom{k+n-1}{k}$.

Example: Consider the set of all weak compositions of 4 with three parts. For each partition, we collect the compositions with the same parts in the same column at the right.

310	400	220	211
301	040	202	121
013	004	022	112
031			
103			
130			

By the formula, we count $\binom{4+3-1}{4}$ = 15 compositions. The count for the first column is six, since we are ordering three distinct numbers. The remaining counts equal the binomial coefficient $\binom{3}{2}$ = 3; we count words of length three with letters x and y, with x appearing twice.

Exercise: Repeat the previous example for all weak compositions of 5 with three parts.

To proceed, it will be helpful, but not necessary, to review the **trinomial coefficient** (or multinomial coefficient with three parts) in the installment "Compositions and Divisors" (see Volume 16, Number 3). In any case, try to work out the following exercise by hand.

Exercise: List all weak compositions of 5 with four parts.

Let's count these compositions by rearranging the entries of the partitions.

Definition: The trinomial coefficient $C(a,b,c) = \dfrac{(a+b+c)!}{a!b!c!}$.

Recall that $C(a, b, c)$ counts the number of words of length $a + b + c$ such that x (resp., y and z) occurs a times (resp., b and c times).

Returning to weak compositions of 5 with four parts, we first denote the corresponding partitions

[1] This is the 14[th] installment in a series that began in Volume 15, Number 3.
[2]This content is supported in part by a grant from MathWorks.

$$5000, \quad 4100, \quad 3200, \quad 3110, \quad 2210, \quad 2111$$

and then count the compositions using the binomial or trinomial coefficient when needed:

$$4 + 12 + 12 + 12 + 12 + 4 = 56 = \binom{5+4-1}{5}.$$

For instance, to count the compositions associated with the partition 3110, we substitute the word *xyyz*. Since y appears twice and x and z appear once, we calculate $C(1, 2, 1) = \frac{4!}{1!2!1!} = 12$.

For longer words, we generalize as follows.

Definition: The **multinomial coefficient** with n parts $C(a_1, a_2, \ldots, a_n) = \frac{(a_1 + \ldots + a_n)!}{a_1! \cdots a_n!}$.

Theorem: The multinomial coefficient $C(a_1, \ldots, a_n)$ counts the number of words in n letters of length $a_1 + \ldots + a_n$ with a_1 x's, a_2 y's, and so on.

Proof: Suppose the theorem is true with $n - 1$ parts. That is, $C(a_2, \ldots, a_n)$ is the number of such words of length $a_2 + \ldots + a_n$. We insert a_1 copies of x into such a word in two steps: First, we choose a_1 positions for x from the total $a_1 + \ldots + a_n$ positions. Then, we fill the remaining positions with one of the $C(a_2, \ldots, a_n)$ words in $y, z, \ldots$. By the Matching Rule, we obtain

$$\binom{a_1 + \ldots + a_n}{a_1} C(a_2, \ldots, a_n) = \frac{(a_1 + \ldots + a_n)!}{a_1!(a_2 + \ldots + a_n)!} \cdot \frac{(a_2 + \ldots + a_n)!}{a_2! \cdots a_n!} = C(a_1, \ldots, a_n). \quad \square$$

Example: The number of weak compositions corresponding to the partition 43322111 of 17 with eight parts is

$$C(1, 2, 2, 3) = \frac{8!}{1!2!2!3!} = 1680.$$

Exercise: Calculate the number of weak compositions of 17 corresponding to the partitions 43222211 and 44321111. What do these partitions have in common?

Exercise: Calculate $C(1, 1, \ldots, 1)$, $C(k - n + 1, 1, \ldots, 1)$, $C(k, 0, \ldots, 0)$, and $C(a_1, \ldots, a_{n-1}, 0)$. Interpret the corresponding words.

Now that we can count the compositions corresponding to a given partition, we have

Theorem: The number $\binom{k+n-1}{k}$ of weak compositions of k into n parts equals the sum of the multinomial coefficients $C(a_1, \ldots, a_n)$, where $a_1a_2\ldots a_n$ ranges over all partitions of k with n or fewer parts.

Example: Consider the weak compositions of 5 with five parts. The corresponding partitions and counts are

$$50000\ \ 41000\ \ 32000\ \ 31100\ \ 22100\ \ 21110\ \ 11111$$

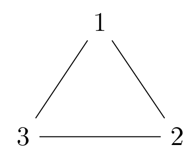
$$5 + 20 + 20 + 30 + 30 + 20 + 1 = \binom{5+5-1}{5} = 126.$$

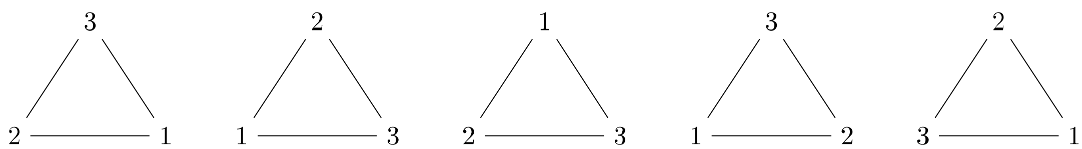
Exercise: Redo the previous example for weak compositions of 6 with four, five, and six parts.

The preceding analysis is common in mathematics and interpreted mathematically with **group theory**. We have a set of interest with a large set of symmetries, we use these symmetries to break the set into smaller sets, and we identify each smaller set by a distinguished element. If we have good intuition for the symmetries, then our task of understanding the larger set reduces to the task of understanding the distinguished elements.

While we only work with groups of permutations in this installment, a very different type of group is found in the four-part Fermat's Little Theorem series (see Volume 6, Numbers 1-4) and the Summer Fun problem set "The Gauss-Wilson Theorem" (Volume 6, Number 5). When we define groups below, it will be enough to keep in mind the following example.

Example: Let G be the set of symmetries that preserve the isosceles triangle at right. Let X be the set of vertices $\{1, 2, 3\}$. If we apply an element g of G to the triangle, the symmetry is entirely described by the final position of the vertices, and we can represent g by the target triangle. By the Matching Rule, there are 6 symmetries; there are three choices for the new position of vertex 1, and two choices remain for vertices 2 and 3.





Exercise: Except for the symmetry that fixes all vertices, each triangle represents either a rotation about the center of the triangle or a reflection across an axis through one vertex. Describe each symmetry as a motion of the triangle. Which triangles have vertices fixed under the symmetry?

To describe each symmetry, the triangle is not entirely necessary. Each symmetry of the triangle corresponds to a permutation of the vertices, and vice versa. For example, consider the clockwise rotation by 120° expressed by the second triangle; the symmetry permutes the vertices as follows:

$$1 \to 2, \qquad 2 \to 3, \qquad 3 \to 1.$$

We have several ways to denote permutations in a more concise manner:

1. **One-line notation:** Organize the vertex data into columns as follows: $\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$. If we assume the top line is ordered, then the bottom line contains all information of the permutation. This permutation is represented by 231 in one-line notation.
2. **Function notation:** As an operation with inputs and outputs, permutations may be represented as functions: $f_{231}(1) = 2, f_{231}(2) = 3, f_{231}(3) = 2$. Of course, each input has a single output, and vice versa.

Exercise: List all symmetries of the triangle as permutations in one-line and function notation.

Exercise: Express the permutations 21435 and 23451 as functions.

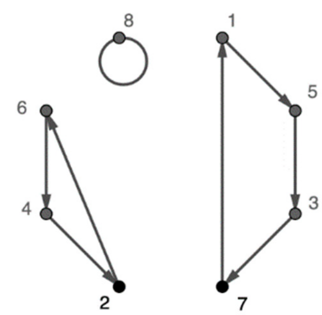
3. **Cycle notation:** Before describing the third notation, let's motivate with an example. One important theme of this series has been the iteration of operations to reveal structure. Let's see what happens if we repeatedly apply the same permutation to a digit.

Example: Consider the permutation 56723418 in one-line notation. If we repeatedly apply this permutation to 1, we obtain the loop 1 → 5 → 3 → 7 → 1. If we think of 1 as the initial position of a bug that travels with each application of the permutation, then the loop property is intuitively true. Since there are only finitely many destinations, the bug must eventually return to a position it once visited before. Because the permutation instructions are fixed, the bug will continue to travel in a loop.

Exercise: Prove that the bug is already in the loop from the start.

We'll prove this later for general situations using group theory. The other loops in this permutation are 2 → 6 → 4 → 2 and 8 → 8.

The permutation is entirely described by these loops, or **cycles.** To emphasize the cycle structure, we instead denote 56723418 as (1537)(264)(8), or (1537)(264) if missing values are understood to be fixed under the permutation. To clarify, entries in a loop are collected in parentheses and ordered so that each entry points to the next entry in the same parentheses; the last entry points to the first entry. This permutation is summarized by the **cycle diagram** at the right.



Exercise: Express each permutation of {1, 2, 3} in cycle notation, and draw the corresponding cycle diagrams.

Exercise: Draw the cycle diagrams for 21453, 21435, and 23451.

Cycle notation is not unique, but the number and size of the cycles are fixed for a given permutation. For instance, (123) = (231) = (312) and (123)(45) = (45)(123).

Example: The possible cycle structures for permutations of {1, 2, 3, 4, 5} are

$$(xxxxx),\ (xxxx)(x),\ (xxx)(xx),\ (xxx)(x)(x),\ (xx)(xx)(x),\ (xx)(x)(x)(x),\ (x)(x)(x)(x)(x).$$

Exercise: In each cycle structure in the example, replace the five x's with the numbers 12345, and draw the cycle diagrams for these permutations.

Exercise: Find the possible cycle structures for permutation lengths 4, 6 and 7, and repeat the previous exercise. Do you recognize a formula for the number of cycle structures? If not, see the installment "Generating Functions for Partitions" (Volume 16, Number 5).

Exercise: Count the cycle structures for permutations where all cycles have length 1 or 2. Conjecture and prove a formula. For instance, for permutations of length 5, we have three types:

$$(xx)(xx)(x),\ (xx)(x)(x)(x), \text{ and } (x)(x)(x)(x)(x).$$

Again, see the installment noted in the previous exercise.

Next, we can define a multiplication of permutations by applying one symmetry after another. In function notation, this multiplication corresponds to composition of functions, where the first symmetry corresponds to the inside function. Since permutations are one-one and onto, their compositions are also. That is, the composition of two permutations is another permutation.

Example: For the triangle, consider what happens if we follow a rotation by 120° clockwise by the reflection that fixes the vertex 1. In one-line notation, the corresponding permutations are 231 and 132, so we calculate $1 \to 2 \to 3$, then $2 \to 3 \to 2$, and finally $3 \to 1 \to 1$. This composition gives the reflection that fixes the vertex 2, or 321 in one-line notation.

Exercise: Satisfy yourself that the previous example, in function form looks like: $f_{132} \circ f_{231} = f_{321}$.

Exercise: Compose the permutations 213 and 312 in either order. Repeat for 21435 and 23451. Does the order of multiplication matter?

Exercise: Compose the permutation 123 with both 132 and 312 in either order. Describe these compositions in terms of the triangle.

Exercise: Express 23451 and 51234 as functions, and compose 23451 with 51234. Then compose 23451 with itself repeatedly until a pattern emerges.

We now have seen an example of all the machinery needed to define a **group**.

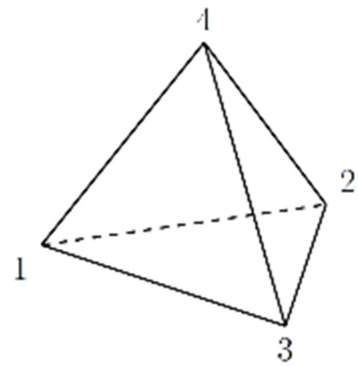
Definition: A **group** $(G, \cdot)$ is a nonempty set G with multiplication $g \cdot h$ satisfying the following three properties:

- **identity:** there is an element 1 such that $g \cdot 1 = 1 \cdot g$ for all g in G,
- **associativity:** for all g, h, k, in G, $(g \cdot h) \cdot k = g \cdot (h \cdot k)$, and
- **inverse:** for each g, there is an h such that $g \cdot h = h \cdot g = 1$. We denote h by g^{-1}.

Intuitively, the elements of the group can be thought of as rearranging the elements of some set X, although the definition does not require such a set X.

In the example above, the six permutations of $X = \{1, 2, 3\}$ form the group S_3 with multiplication given by composition. In one-line notation, the identity element is the permutation 123, and, for a given permutation in function notation, the inverse element is the inverse function. To calculate the inverse in one-line notation $a_1a_2a_3$, write 1 in position a_1, 2 in position a_2, and 3 in position a_3. For instance, the inverse of 231 is 312 and the inverse of 132 is 132. With 1, 2, and 3 as the labels of the vertices of a triangle, we see that rotation by 120° clockwise inverts rotation by 120° counter-clockwise and reflections are their own inverse.

Exercise: Find all permutations of {1, 2, 3, 4} in one-line notation. Find the inverse of each permutation. Express each permutation as a symmetry of the tetrahedron at right. These permutations form the group S_4.



Exercise: If we label the vertices of a square clockwise 1, 2, 3, and 4, we can express its symmetries as permutations of {1, 2, 3, 4}. Find the symmetries of this square in one-line notation. These permutations form the group D_8.

Exercise: Interpret each of the defining properties of a group in the context of a group of symmetries.

Exercise: Prove that subtraction is not associative: $(a - b) - c \neq a - (b - c)$.

Exercise: Prove that the identity element 1 of a group is unique. That is, if 1_1 and 1_2 are identity elements in a group G, then $1_1 = 1_2$.

Exercise: Prove that inverses are unique. That is, if h and k are both inverse to g, then $h = k$.

Next, we use the group properties together to derive an important result for solving equations with group elements.

Theorem (Cancellation Law): Suppose g, h, and k are elements of the group G. If $g \cdot h = k$ then $h = g^{-1} \cdot k$.

Proof: We multiply by g^{-1} on the right of both sides of the equation $g \cdot h = k$, to get

$$g^{1} \cdot (g \cdot h) = g^{-1} \cdot k.$$

By the properties of associativity, inverse, and identity, we have

$$g^{-1} \cdot (g \cdot h) = (g^{-1} \cdot g) \cdot h = 1 \cdot h = h. \quad \square$$

Exercise: If g, h are elements of the group G, prove that $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$.

Finally, we consider properties that follow from finiteness conditions. These properties hold for groups with a finite number of elements.

Definition: If g is an element of a group and $n > 0$, define $g^n = g \cdot g \cdots g$ (n factors of g). If there is a smallest positive n such that $g^n = 1$, we define n to be the **order** of g and denote this by $|g|$. We also say that g has **finite order**. In a similar manner, we denote the number of elements in the group G by $|G|$.

Examples: The identity element 1 has order 1. Also, $|231| = 3$ and $|132| = 2$ in S_3.

Exercise: Find the orders of each element in S_4 and D_8. Interpret the order of each element in terms of symmetries of the tetrahedron and the square. Do you see a relationship between the orders of group elements and the order of G?

If an element g has finite order n, then $g^n = 1$ and the cancellation law leads to the next formula.

Formula for inverses: $g^{-1} = g^{n-1}$.

Example: In S_3, $f_{231}{}^{-1} = f_{231}{}^2 = f_{312}$ and $f_{132}{}^{-1} = f_{132} = f_{132}$.

Exercise: Verify the inverse formula for all elements of S_4 and D_8.

Exercise: If g has order 2, prove that $g = g^{-1}$.

Exercise: If every element of G has order 2, prove that $g \cdot h = h \cdot g$ for all g, h in G.

Definition: We say G is **abelian** (or **commutative**) if $g \cdot h = h \cdot g$ for every g, h in G.

Exercise: Prove that the following examples are abelian groups:

1. the integers under the operation of addition,
2. the non-zero real numbers under usual multiplication,
3. the complex numbers with absolute value 1 under complex multiplication,
4. the integers modulo n for some fixed integer $n > 1$ under modular addition, and
5. the modular units $U_n \equiv \{k \mid \text{there exists } m \text{ such that } n \text{ divides } km - 1\}$ under modular multiplication.

See the series on Fermat's Little Theorem for part 4 and the "The Gauss-Wilson Theorem" problem set for part 5.

Note that it is probably clearer to refer to the group multiplication as the group "operation," as natural examples use both addition and multiplication. In those cases, we need to clarify which operation is used to define the group.

Exercise: Fix $n > 2$. Let P_n be a regular polygon with n sides. How many symmetries does P_n have? Prove that the set of symmetries, denoted by D_{2n}, forms a group under composition. Is the group D_{2n} abelian? List all symmetries in D_{10} and D_{12} as permutations in one-line and cycle notation.

In the next installment, we'll explore the notion of a **group action** to formalize symmetry as a group property and apply this machinery to the original example with permutations and compositions. We'll also say more about the following problems.

Exercise: How many elements of order 2 are in S_4 and S_5? List these elements in cycle notation; recall the two types of redundancies for a permutation in cycle structure noted above. What do they have in common? Then, describe the elements of order 2 in S_6.

Exercise: Count the permutations for each cycle structure in S_4 and S_5. Can you do it without listing them? How far can you get with S_6?

Cubics, Part 2

by Lightning Factorial | edited by Jennifer Sidney

In the spirit of figuring things out, we asked Lightning Factorial to try to find a formula for the roots of a cubic equation in terms of its coefficients. The cubic formula, like the quadratic formula, is well known and can readily be looked up. But trying to figure out something yourself can take you on a journey that's far more fun. Let's rejoin Lightning's cubic math adventure!

Last time, I tried to look at a certain class of cubic equations (with real coefficients) that are known to have a unique real root. I tried to find that unique real root, but I ended up discovering that the real part of its nonreal roots (as both of its nonreal roots have the same real part) is, itself, the root of a cubic equation with real coefficients.

Specifically, if the cubic equation $x^3 + bx^2 + cx + d = 0$ has a unique real solution, then the real part of its nonreal solutions is the negative of the real root of the cubic equation

$$t^3 - bt^2 + (c + b^2)t/4 + d/8 = 0.$$

(It's the *negative* of the real root because the real root of this equation is the value of t that will translate the roots to the right so that the nonreal roots become pure imaginary, and the real part of the nonreal roots is the negative of that.) I thought that was kind of neat, even though it doesn't help me solve the original cubic equation. In fact, I'd like to test it with an actual example. Let's consider the cubic with roots $1 + i$, $1 - i$, and 0, that is, the cubic

$$x(x - (1 + i))(x - (1 - i)) = x^3 - 2x^2 + 2x,$$

so $b = -2$, $c = 2$, and $d = 0$. Since the real part of its nonreal roots is 1, we should see that -1 is a root of the cubic equation

$$t^3 + 2t^2 + (2+(-2)^2)t/4.$$

Oh no! It's not, because $(-1)^3 + 2(-1)^2 + 6(-1)/4$ is -1/2, not 0! What went wrong?

Please go back to the last installment and try to figure out Lightning's error before reading on.

Going back over my calculations, I see that I dropped a term. I got to the following equation:

$$(3t - b)(3t^2 - 2tb + c) = t^3 - bt^2 + ct - d,$$

but then I simplified incorrectly because I missed a term. The correct simplification is

$$t^3 - bt^2 + (c + b^2)t/4 + (d - bc)/8 = 0.$$

So when $b = -2$, $c = 2$, and $d = 0$, this becomes $t^3 + 2t^2 + 6t/4 + 4/8 = 0$. Gosh, I hope that -1 is a root of this cubic! Let's see: $(-1)^3 + 2(-1)^2 + 3(-1)/2 + 1/2 = -1 + 2 - 3/2 + 1/2 = 0$ … yes!

Unfortunately, the resulting cubic doesn't seem any easier to solve than the original cubic, so it doesn't seem like this approach helps.

I know that in general, if the roots of the cubic $x^3 + bx^2 + cx + d$ are r_1, r_2, and r_3, then

$$\begin{aligned} x^3 + bx^2 + cx + d &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_2r_3 + r_3r_1)x - r_1r_2r_3 \end{aligned}$$

By comparing coefficients, we get Vieta's formulas:

$$\begin{aligned} b &= -(r_1 + r_2 + r_3) \\ c &= r_1r_2 + r_2r_3 + r_3r_1 \\ d &= -r_1r_2r_3 \end{aligned}$$

In the situation I was hoping to solve initially – where $x^3 + bx^2 + cx + d$ is strictly increasing and has a unique real root I called r – the other two roots are of the form $R + Si$ and $R - Si$, where R and S are real numbers and i is the square root of -1, because nonreal roots of polynomials with real coefficients come in complex conjugate pairs. From the first of Vieta's formulas above, that means $2R + r = -b$. If I could solve for the unique real root of $t^3 - bt^2 + (c + b^2)t/4 + (d - bc)/8$, which should be $-R$, I could then find the unique real root of the original cubic by computing $-b - 2R$.

Hm. I can turn a cubic with a root of $-R$ into a cubic with a root of $-b - 2R$ by scaling by 2 and translating (left) by b. If I do that to the cubic $t^3 - bt^2 + (c + b^2)t/4 + (d - bc)/8$, would I get back the original cubic? I'm going to try it and see!

The cubic

$$t^3 - bt^2 + (c + b^2)t/4 + (d - bc)/8$$

has the unique real root $-R$, so the cubic

$$(t/2)^3 - b(t/2)^2 + (c + b^2)(t/2)/4 + (d - bc)/8$$

will have the unique real root $-2R$. This cubic simplifies to $t^3/8 - bt^2/4 + (c + b^2)t/8 + (d - bc)/8$. Thus the cubic

$$(t + b)^3/8 - b(t + b)^2/4 + (c + b^2)(t + b)/8 + (d - bc)/8$$

will have the unique real root $-b - 2R$. After a bit of algebra (which I'll spare you, but please double-check for yourself!), this simplifies to

$$t^3/8 + bt^2/8 + ct/8 + d/8,$$

which is just the original cubic divided throughout by 8 (ignoring the change in variables from x to t)!

In other words, if I let $p(x) = x^3 + bx^2 + cx + d$ and $q(t) = t^3 - bt^2 + (c + b^2)t/4 + (d - bc)/8$, then $p(x) = 8q((x + b)/2)$, or, flipping it around, $q(x) = p(2x - b)/8$.

Unfortunately, scaling and shifting the cubic polynomial left or right preserves the quality of being a cubic polynomial!

I guess by translating horizontally, we could transform the cubic into one that has no quadratic term, thereby obtaining a cubic whose roots sum to 0. That would be effected by translating to the right by $b/3$. We'd get the cubic

$$(x - b/3)^3 + b(x - b/3)^2 + c(x - b/3) + d = x^3 + (c - b^2/3)x + 2b^3/27 - bc/3 + d.$$

That means that if I can solve cubic equations that lack a quadratic term, I would be able to solve any cubic equation.

Actually, that's essentially the same idea behind "completing the square" for quadratic equations. There, one translates the parabola horizontally and eliminates the linear term, resulting in a quadratic equation that can readily be solved. Unfortunately, with cubic equations, elimination of the quadratic term doesn't seem to make it easier to solve!

But we might as well take the simplification. So from here on, let's focus on cubic polynomials of the form $x^3 + cx + d$.

I wonder if there is a translation that does lead to an equation that can be solved more easily. A cubic polynomial without a quadratic term places the 0 of the complex plane at the centroid of the triangle whose vertices are defined by its roots. What if I translate horizontally so that the *circumcenter* of the triangle becomes the 0 of the complex plane? That way, all the roots will be the same distance from the origin, so their magnitude would be the cube root of the absolute value of the constant term; and since one of the roots is real, the cube root of the constant term would be, up to a sign, the real root of the cubic!

Let's try that! Hopefully, the amount we have to translate by in order to place the circumcenter at 0 will be some nice function of the coefficients. If the roots are, again, r, $R + Si$, and $R - Si$, where is the circumcenter? By symmetry, it is on the real axis, so it will be located where the perpendicular bisector of the line segment that connects r and $R + Si$ intersects the real axis. Switching to Cartesian coordinates, the midpoint of the line segment connecting $(r, 0)$ with (R, S) is $((R + r)/2, S/2)$. Since the slope of that line segment is $S/(R - r)$, the slope of the perpendicular bisector must be $(r - R)/S$. Therefore, the equation of the perpendicular bisector is

$$y - S/2 = (r - R)(x - (R + r)/2)/S.$$

This line intersects the real axis when $y = 0$, so I need to solve the equation

$$-S/2 = (r - R)(x - (R + r)/2)/S$$

for x. I get $x = \dfrac{R^2 + S^2 - r^2}{2(R - r)}$. Now the question is whether this quantity can be expressed in terms of the coefficients.

Let's see. From Vieta's formulas, if the quadratic coefficient is 0, we get

$$0 = 2R + r$$
$$c = R^2 + S^2 + 2Rr$$
$$d = r(R^2 + S^2).$$

From the first equation, $r = -2R$. That means that $c = R^2 + S^2 - r^2$, which is the numerator of our expression for the circumcenter! Therefore,

$$x = \frac{R^2 + S^2 - r^2}{2(R - r)} = \frac{c}{2(R - r)} = -\frac{c}{3r}.$$

Oh dear. This last expression means that finding an expression for the circumcenter in terms of the coefficients is basically the same thing as finding an expression for r in terms of the coefficients, which means being able to solve the cubic! This cubic is like a tight clam.

Maybe I should think about what the form of the roots of a cubic can look like. After all, knowing the quadratic formula, we know that the root of a quadratic with rational coefficients must be of the basic form $m + \sqrt{n}$, where m and n are rational numbers. And from this, we can see why it is possible to solve the quadratic equation with a translation of the polynomial by a rational amount. If the roots of a cubic with rational coefficients all had the form $m + \sqrt[3]{n}$, where m and n are rational numbers, then we should also be able to solve cubic equations by translating by a rational amount. Since that doesn't seem possible, it suggests that some cubic equations with rational coefficients will have roots that are not expressible in that form.

However, I'm not sure what form to use. Perhaps $\sqrt[3]{m} + \sqrt[3]{n}$, where m and n are rational numbers? Or perhaps $\sqrt{m} + \sqrt[3]{n}$?

Since no particular form stands out to me, maybe I can simply try $u + v$, and by trying that, something will inform me what forms u and v can be. That is, I'm hoping that the roots of a cubic equation can be split into a sum where each summand can be found either as the root of a cubic equation that is readily solvable, or perhaps even a quadratic equation.

Here goes. If $u + v$ is a root of $x^3 + cx + d$, then

$$(u + v)^3 + c(u + v) + d = 0.$$

Expanding this out, I get

$$u^3 + 3u^2v + 3uv^2 + v^3 + cu + cv + d = 0.$$

I can also write this as

$$u^3 + 3uv(u + v) + v^3 + c(u + v) + d = 0,$$

or $u^3 + (3uv + c)(u + v) + v^3 + d = 0$.

That $(3uv + c)(u + v)$ term sure complicates matters. If that weren't there, the equation would just be $u^3 + v^3 + d = 0$, and I can find many solutions to this equation. One could just plug in any value for u and solve the equation for v.

Hold on! What would happen if I simply *declare* $3uv + c$ to be equal to 0? Then, even though there are infinitely many solutions to the equation $u^3 + v^3 + d = 0$, only some of them would satisfy $3uv + c = 0$ as well, and any solution (u, v) that satisfied both equations would give the solution $u + v$ for the cubic equation $x^3 + cx + d = 0$!

That's exciting! I've got to try it!

So if $3uv + c = 0$, then $v = -c/(3u)$. The equation

$$u^3 + v^3 + d = 0$$

could then be rewritten

$$u^3 - \frac{c^3}{27u^3} + d = 0,$$

which is a quadratic equation in u^3! We can solve for u^3 using the quadratic formula!

I think this opens the clam!

Thus, to find the roots of a cubic, we first divide it by its lead coefficient to get a cubic with lead coefficient equal to 1 of the form $x^3 + bx^2$ + (some constant)x + (some other constant). Then we perform a horizontal translation by $-b/3$ to obtain a cubic of the form $x^3 + cx + d$. Next, we solve the quadratic equation $U^2 + dU - c^3/27 = 0$. Let's let u be the cube root of a solution to this quadratic equation, and let $v = -c/(3u)$. Then $u + v$ should be a solution to $x^3 + cx + d = 0$. We can then obtain a solution to the original cubic equation by adding $b/3$ to $u + v$!

It's strange because there are generally two solutions to the quadratic, and each solution will typically yield three cube roots. Wouldn't that produce *six* solutions to the cubic equation? Something fishy is going on here. There can't be six different solutions to a cubic equation! That must mean that different choices of square and cube roots will yield the same value of $u + v$. But that opens up the possibility that they might all yield the same value, in which case we'll have only found one solution of the cubic equation. But I guess that's okay, because if you can find one solution to a cubic equation, you can reduce it to a quadratic equation and solve for the remaining two roots with the quadratic formula.

In any case, I really need to sort out exactly what's going on here. Also, what do we do if u happens to be 0? In that case, the equation $3uv + c = 0$ won't have a solution unless c is also 0. Lots more to think about!

Can you sort out this situation before
Lightning Factorial's next installment?

Modeling Power Transmission Lines

by Cecilia Esterman

These problems give a basic idea of the kinds of problems I analyze in my work at Avangrid.

1. A town of a hundred people is situated at the end of a long road from a power station. A series of n transmission towers are placed with wires strung between them along the road to the town. On any given day, the probability that one of these towers fails is p. If any of the towers fail, the town loses power. Assuming that the towers fail independently of each other, what is the probability that the town lose power on any given day?

For Problem 2-5, there is a power station that supplies power to two towns, towns A and B. For town A, there are two transmission towers used to get power there, one old and one new. For town B, there are four transmission towers between the town and the power station, two old and two new. In any given year, the probability that a new tower fails is 1/6, whereas the probability that an old tower fails is 1/2.

2. In any given year, what is the probability that town A loses power? What is the probability that town B loses power?

3. Suppose there are 100 people who live in town A and 100 people who live in town B. The power station is able to replace one of the old transmission towers with a new one. To minimize the expected number of customers who will lose power in any given year, should an old tower be replaced on the line that services town A or town B?

4. Now suppose there are only 40 people who live in town A and 100 people who live in town B. To minimize the expected number of customers who will lose power in any given year, should an old tower be replaced that services town A or town B?

5. For what ratio of populations of town A to town B does it not matter which old tower is replaced (because replacing any old tower gives the same reduction in customer power losses)?

For Problems 6-9, there's a distribution network stretching out from the substation passing by N houses. Each house has a population of 5. Between each pair of consecutive houses, there is a distribution pole, and there's a distribution pole between the substation and the first house. In this scenario, if a distribution pole fails, all the houses farther away from the substation lose power, but the other houses do not. On any given day, the probability that a distribution pole fails is p, and each distribution pole fails independently of the others.

6. What it the probability that the kth house from the substation loses power on any given day?

7. How many customers are expected to lose power on any given day?

8. If $N = 2$, what must p be so that only 1% of all the people are expected to lose power on any given day?

9. If $N = 3$, what must p be so that only 1% of all the people are expected to lose power on any given day?

Member's Thoughts

Deriving the Quadratic Formula from Scratch

by Ken Fan

In *Discovering the Quadratic Formula*[1] and *The Quadratic Formula, Revisited*,[2] Addie Summer and Lightning Factorial explained how they figured out the quadratic formula from scratch. As you study math, try to figure things out before reading solutions. With patience, you will discover that you can figure out quite a bit! In fact, last fall at Girls' Angle, member Viola Shephard tried her hand at deriving the quadratic formula using only her mind. She succeeded!

That is, she found the solutions to the equation $ax^2 + bx + c = 0$, where x is the unknown to be solved for, and a, b, and c are constants.

Now, any way of solving that equation will ultimately result in the famous quadratic formula. However, there are many ways of getting there, and Viola's method is subtly different from the standard approaches that can be found in textbooks or in the two articles mentioned above, so it's worth a look. Here's what she did:

She noted that she could solve quadratic equations of the special form $(mx + n)^2 = p$, where m, n, and p are constants. This expands to $m^2x^2 + 2mnx + n^2 = p$.

She then asked: How closely can I make this quadratic equation look like $ax^2 + bx + c = 0$?

She saw that if she set $m = \sqrt{a}$, then the coefficients of the x^2 terms will be the same. And then she noted that she could let $n = b/(2m) = b/(2\sqrt{a})$ to make the coefficients of x be the same. Getting two out of the three coefficients to be equal is pretty good! Indeed

$$(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = ax^2 + bx + \frac{b^2}{4a}$$

differs from $ax^2 + bx + c$ only by the constant $c - b^2/(4a)$. Thus, Viola found that

$$ax^2 + bx + c = (\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 + c - \frac{b^2}{4a}$$

But the equation $(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 + c - \frac{b^2}{4a} = 0$ is exactly of the kind that Viola noted she could solve! Just add $b^2/(4a) - c$ to both sides, take square roots, and then isolate x in the resulting linear equation. Voila, the quadratic formula!

How natural and beautiful a derivation of the quadratic formula! Congratulations, Viola!

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[1] See pages 21-23 of Volume 10, Number 2 of this Bulletin.
[2] See pages 12-13 of Volume 11, Number 1 of this Bulletin.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 34 - Meet 5 March 7, 2024	Mentors:	Elisabeth Bullock, Jade Buckwalter, Serina Hu, Shauna Kwag, Gautami Mudaliar, Hanna Mularczyk, AnaMaria Perez, Vievie Romanelli, Swathi Senthil, Padmasini Venkat, Jing Wang

If you like different number systems and haven't heard of p-adic numbers yet, check them out! Briefly, let p be a prime number. Consider sequences of integers $\{a_k\}$ with the property that for any positive integer m, there exists N such that $a_k - a_j$ is divisible by p^m whenever k and j exceed N. We define an equivalence relation on these sequences by declaring that two such sequences $\{a_k\}$ and $\{b_k\}$ are equivalent if and only if for any positive integer m, there exists N such that $a_k - b_k$ is divisible by p^m for all $k > N$. The p-adic numbers are the equivalence classes of such sequences. Addition and multiplication of such sequences are defined component-wise. Unlike in the integers, rational numbers, or the real numbers, in this number system, for some prime numbers p, the equation $x^2 = -1$ has solutions. Can you figure out for which prime numbers p the square root of -1 exist? (In the p-adic numbers, -1 corresponds to the sequence where every term is -1.)

Session 34 - Meet 6 March 14, 2024	Mentors:	Elisabeth Bullock, Jade Buckwalter, Clarise Han, Gautami Mudaliar, AnaMaria Perez, Swathi Senthil, Padmasini Venkat, Jing Wang

Thinking about Fibonacci numbers modulo n, for some fixed integer $n > 1$ is a rich source of mathematics!

Also, some members worked through some of the problems in the book *The Mathematics of Secrets: Cryptography From Caesar Ciphers to Digital Encryption* by Joshua Holden. If you're curious to know how two people can set up a way of sending secret messages *without* having to have a private meeting to discuss how to do it (that is, create a way of sending secret messages while also assuming that their conversations are being overheard by people that they do not want to be able to decipher their messages), check out this book. (Or, try to think of a way to do it yourself!)

Session 34 - Meet 7 March 21, 2024	Mentors:	Elisabeth Bullock, Jade Buckwalter, Serina Hu, Shauna Kwag, Gautami Mudaliar, AnaMaria Perez, Swathi Senthil, Padmasini Venkat, Jane Wang, Jing Wang, Saba Zerefa

How many ways can you think of to show that the vertex of the parabola $y = ax^2 + bx + c$ occurs where $x = -b/(2a)$? Can you show it without using calculus? Speaking of quadratic polynomials, how many ways can you think of to derive the quadratic formula? (See *Member's Thoughts* on page 26 to see how member Viola Shephard found her own personal way of deriving the quadratic formula!)

Session 34 - Meet 8
April 4, 2024

Mentors: Elisabeth Bullock, Gautami Mudaliar, AnaMaria Perez, Swathi Senthil, Alaina Silverman, Padmasini Venkat, Jane Wang, Jing Wang, Dora Woodruff, Saba Zerefa

The interconnections in mathematics are so intricate and far reaching. Frequently, at Girls' Angle, two independent groups of girls will somehow find themselves working on the same thing, having arrived there from completely different directions. At today's meet, the Apollonian gasket sprouted up in this way. If you don't know what the Apollonian gasket is, draw three circles in such a way that each is tangent to the other two. Now, add circles to your picture in such a way that each circle you add is tangent to three circles already drawn, and so that whenever the circle touches another, the two touching circles must be tangent to each other. You'll find that you can continue this process indefinitely (and in many different ways). The resulting pattern of circles is an example of an Apollonian gasket.

Session 34 - Meet 9
April 11, 2024

Mentors: Elisabeth Bullock, Jade Buckwalter, Bridget Li, Gautami Mudaliar, AnaMaria Perez, Swathi Senthil, Alaina Silverman, Dora Woodruff, Saba Zerefa

How would you generalize Pascal's triangle to three or more dimensions?

Session 34 - Meet 10
April 25, 2024

Mentors: Alexandra Fehnel, Bridget Li, Gautami Mudaliar, AnaMaria Perez, Vievie Romanelli, Dora Woodruff, Saba Zerefa

Visitor: Cecilia Esterman, Avangrid

Cecilia Esterman is a recent graduate of MIT who now works at Avangrid, a clean energy company. She mentored at Girls' Angle while she was at MIT. At Avangrid, she uses math to model real-world problems. Her analyses are used to:

- Quantify the benefit of replacing an asset
- Quantify the benefit of adding or upgrading an asset
- Analyze which projects are worth pursuing

To illustrate, she engaged us with a probabilistic simulation of power transmission line failure. If a transmission line fails, customers lose electricity. Using a mathematical model, she can analyze the probability that people lose power and this can help power companies determine how to allocate resources.

For a sampling of problems that give a flavor of the kind of work she does, see Cecilia's problem set *Modeling Power Transmission Lines* on page 25.

Another example of the kind of problem she models with mathematics can be understood by imagining that you are a newsvendor who sells newspapers. If you don't buy enough newspapers to meet demand, you will lose out on revenue. However, if you buy too many papers, you will have incurred a cost due to purchasing papers that you can no longer sell at price. (You can't sell yesterday's paper for full price!) How can you determine the optimal number of newspapers to purchase in order to maximize profits?

Calendar

Session 33: (all dates in 2023)

September	14	Start of the thirty-third session!
	21	
	28	Support Network Visitor: Isable Vogt, Brown University
October	5	
	12	
	19	
	26	
November	2	
	9	
	16	
	23	Thanksgiving - No meet
	30	
December	7	

Session 34: (all dates in 2024)

February	1	Start of the thirty-fourth session!
	8	
	15	
	22	No meet
	29	
March	7	
	14	
	21	
	28	No meet
April	4	
	11	
	18	No meet
	25	Support Network Visitor: Cecilia Esterman, Avangrid
May	2	
	9	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

____-

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- □ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- □ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high-level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: ________________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax-free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ___________________________________