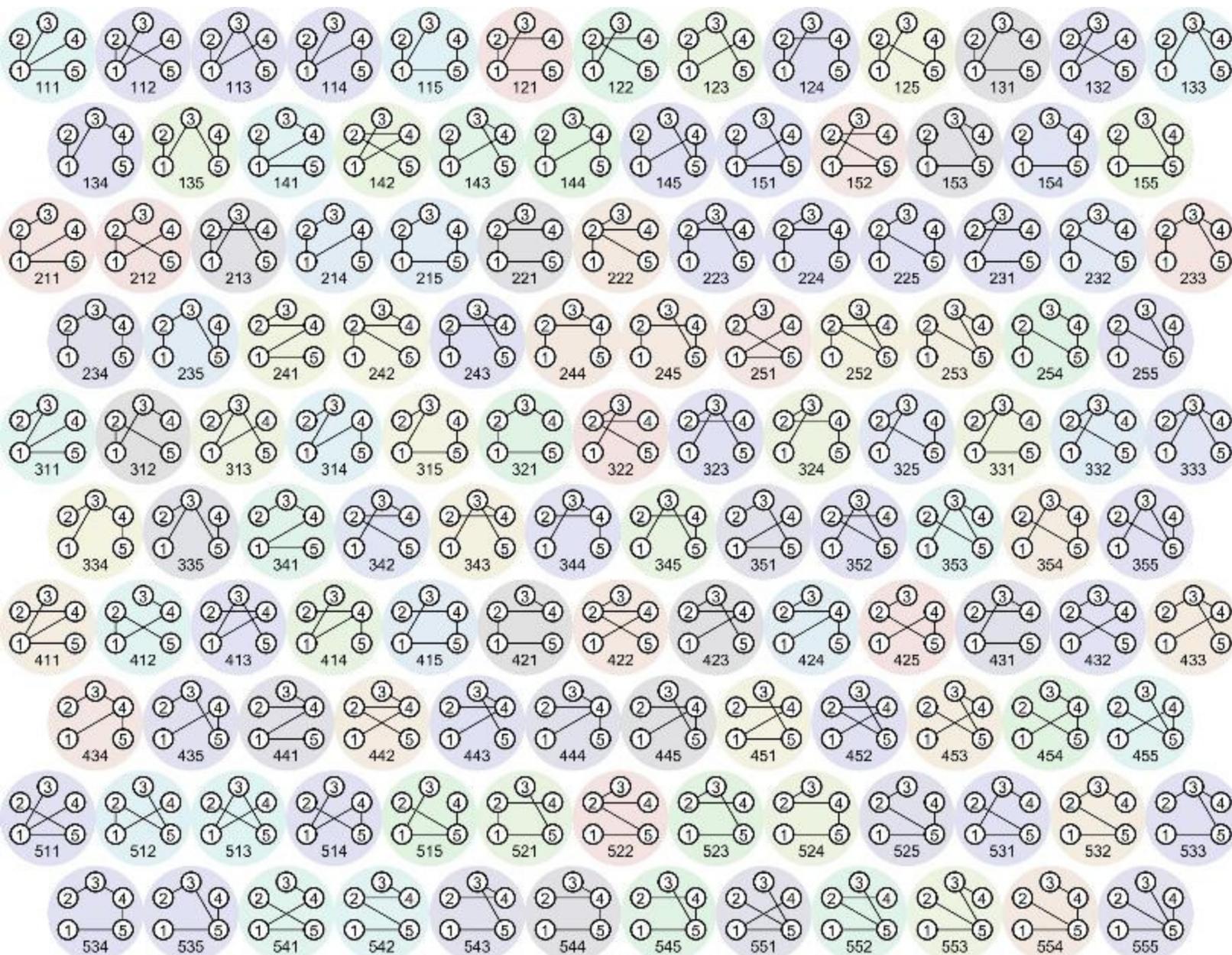


# Girls' *Angle* Bulletin

August/September 2022 • Volume 15 • Number 6

*To Foster and Nurture Girls' Interest in Mathematics*



An Interview with Gloria Marí Beffa: Gender and Math  
Fibonacci Numbers and Multiset Counting  
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## From the Founder

If math amazes you, but you think, "I could never do that!" I get it. But there's no sense in comparing initial attempts to do math to what's been done. So don't be deterred! Beautiful ideas are all born unrefined and require time and care to emerge. - Ken Fan, President and Founder

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## Girls' Angle Bulletin

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## Girls' Angle: A Math Club for Girls

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*

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On the cover: *Labeled 5-Trees* by C. Kenneth Fan. See Laura Pierson's Summer Fun solutions on page 20.

# An Interview with Gloria Marí Beffa: Gender and Math

This is a second interview with Gloria Marí Beffa, Professor of Mathematics at the University of Wisconsin-Madison. In our earlier interview with Professor Marí Beffa, she made some tantalizing remarks pertaining to gender and math. We asked her if she would be willing to do a second interview focused on that topic. She graciously agreed. Here, we present this second interview with Gloria Marí Beffa on gender and math.

**Ken:** Some scholars, like Steven Pinker,<sup>1</sup> suggest that the low representation of women in math does not reflect gender bias, but, rather, a biological fact that girls just don't like math as much as boys. What are your thoughts on this matter?

**Gloria:** They ignore reality. Many years ago, I learned that Icelandic math was actually dominated by women – unfortunately I cannot remember where I learned that – and Italian math also has a significantly higher number of female faculty. When I was an undergraduate in Spain I had a significant number of female professors, while I had none (literally zero) while in graduate school in the US where they had only one female faculty (out of over 70 faculty members) in a subject I was not interested in. Unless US women are genetically predisposed to not like math, versus Icelandic and Italian females, that conclusion is not sustained by any

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<sup>1</sup> Steven Pinker is the Johnstone Family Professor at the Department of Psychology at Harvard University. Some of his views on gender and math may be found in the transcript of a debate that he had with Harvard

A relevant excerpt from our first interview with Professor Marí Beffa.

**Ken:** I love the idea of Girls Math Night Out! Of course, I also started a math program for girls. Yet, I've been surprised at the resistance to the notion of an all-girl math program. Some have told me that it sends the wrong message that "girls need extra help," and some have assumed that we "water down math because we teach girls." Did you encounter such resistance with Girls Math Night Out! When you encounter resistance, how do you respond?

**Gloria:** I always ignored all of that. The best answer is to prove them wrong. Some of the girls in the program went on to be engineers and math majors. That, and the experience with the groups is all I needed to know.

...

**Ken:** Do you think there is gender bias against women in math? If so, what can we do to eradicate it?

**Gloria:** Yes, absolutely, not only against cis women, but also against trans women and other genders. I think I would need another interview to answer the second part of this question.

**Ken:** I'd be super interested in such a second interview!

reasonable data. I think the differences in math gender representation across the globe must be much more strongly linked to cultural differences, which is the conclusion of many other studies. Plus, anyone who knows anything about stereotyping knows that female negative STEM stereotypes are rampant in the American culture, and they have very predictable consequences. I think the literature is quite clear at this point about the origins of a large portion of these differences, and if there are other factors their impact is probably quite small compared to that of our cultural background.

**Ken:** Many of these issues lead to questions of what is biological versus what is cultural? For example, I believe I've sensed an average gender difference in the motivation imparted by math competitions. Even if we accept that this is the case, there's still the question of whether this is due to some biological difference or it's due to

Professor of Psychology Elizabeth Spelke which took place on April 22, 2005 and which was organized by Harvard University's Mind/Brain/Behavior Initiative.

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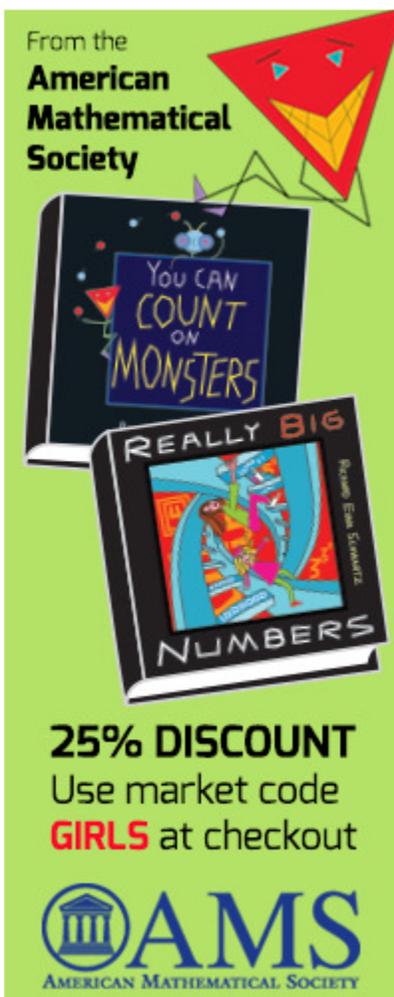
For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Mari Beffa and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

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Thank you and best wishes,  
Ken Fan  
President and Founder  
Girls' Angle: A Math Club for Girls

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The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to <http://www.ams.org/bookstore-getitem/item=MBK-84-90> and use the code “GIRLS” at checkout.

# Fibonacci Numbers and Multiset Counting

by Robert Donley<sup>1</sup>

edited by Amanda Galtman

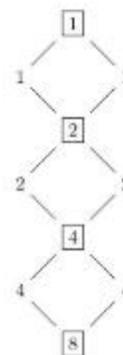
In the previous two parts, we counted walks on the number line and the half line. These counting problems yielded two important types of numbers from combinatorics: **binomial coefficients** and **Catalan numbers**. The walks have no restrictions in the first case, while the Catalan numbers arise from loops that take positions only to the right of the negative numbers.

In this part, we restrict walks to a finite interval to obtain the **Fibonacci numbers**. To derive the famous Binet formula for the Fibonacci numbers, we return to the concepts of generating functions and convolution, and these ideas lead to an interesting aside on multiset counting. We highly recommend that you first read the article “The Fibonacci Function” in Volume 3, Number 1 of this Bulletin; the approach there is much more intuitive and does not use generating functions. (For past issues of the Bulletin, please visit [www.girlsangle.org/page/bulletin.php](http://www.girlsangle.org/page/bulletin.php).)

We recall the process for constructing and counting walks in a bounded interval. We start on the number line at position zero at time zero, and, as each second passes, we move one unit either to the left or to the right with equal chance, as allowed. We call this process a **random walk** (or simply **walk**). Reading left to right, we form a word by recording the move performed at the end of each second; after  $N$  seconds, we obtain a word of length  $N$  with letters  $L$  and  $R$ .

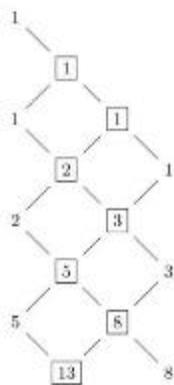
If our walks are restricted to the positions 0 and 1, then the walk merely travels back and forth, and the corresponding word of length  $N$  has pattern  $RLRLRLR\dots$ . Although this case appears somewhat uninteresting, we will see that its generating function is extremely useful.

If we restrict the walk to the positions -1, 0, and 1 and repeat the Pascal’s triangle construction (see figure at right), the central values (at 0) form a **geometric sequence**. As before, these values record the number of loops at time  $2N$ . Note that



- The value at time 0 equals 1,
- The values for odd time  $2N - 1$  are equal, and
- The values for even time  $2N$  have twice the value at time  $2N - 2$ .

In fact, the number of loops at time  $2N$  is equal to  $2^N$ .



Next, we restrict the walk to the positions 0, 1, 2, and 3. If we construct the corresponding grid (see figure at left), we rediscover a familiar sequence: the **Fibonacci numbers**, the first 8 values of which are 1, 1, 2, 3, 5, 8, 13, and 21. The Fibonacci numbers, which we’ll denote by  $F_n$ , can be defined recursively by letting  $F_1 = F_2 = 1$ , and  $F_N = F_{N-1} + F_{N-2}$  for  $N > 2$ . In other words, the first two values are 1, and later terms are given by adding the previous two terms together. In the corresponding grid, entries in each lower row are given by two consecutive Fibonacci numbers. Pascal’s recurrence simply moves the larger entry to the next row; the new entry is the next Fibonacci number.

<sup>1</sup> This content is supported in part by a grant from MathWorks.

**Exercise:** Calculate the first 15 Fibonacci numbers.

Let's find the generating functions for these path counting sequences. If we have a sequence  $a_n$ , its **generating function** is given as a formal sum

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

By "formal," we mean that we're not concerned about whether the series converges. We are using power series essentially as a bookkeeping device, organizing the data into the coefficients so that we may employ addition and multiplication of power series. (Instead of using the notation of power series, we could just work with infinite tuples  $(a_0, a_1, a_2, \dots)$  and define an addition and multiplication on these tuples that correspond to how the coefficients combine in addition and multiplication of power series.)

**Example:** For the walks restricted to positions 0 and 1, the sequence that counts walks is simply the sequence of 1's, so we obtain the formal **geometric series**

$$F(x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}.$$

We treat the second equality formally, that is, we write  $1/(1-x)$  because  $(1-x)F(x) = 1$ . (Check this!)

**Exercise:** Fix  $n \geq 1$ . Express the following generating function as a rational function:

$$F(x) = 1 + x + x^2 + \dots + x^n.$$

**Example:** For the walks restricted to positions 0, 1, and 2, the generating function for counting loops is given by

$$G(x) = 1 + 2x + 4x^2 + 8x^3 + \dots = \frac{1}{1-2x}.$$

In this case, we simply substitute  $2x$  for  $x$  in the previous geometric series.

To find the generating function for the Fibonacci numbers, our main technique will be the general concept of **discrete convolution** (or **Cauchy product**) of sequences, for which Chu-Vandermonde convolution is a special case.

Let  $F(x) = a_0 + a_1x + a_2x^2 + \dots$  and  $G(x) = b_0 + b_1x + b_2x^2 + \dots$ . Observe that the product  $F(x)G(x)$  is the generating function  $c_0 + c_1x + c_2x^2 + \dots$  where

$$c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \dots + a_nb_0.$$

We call the sequence  $c_0, c_1, c_2, \dots$  the discrete convolution of the sequences  $a_n$  and  $b_n$ . If  $a$  is the sequence  $a_0, a_1, a_2, \dots$  and  $b$  is the sequence  $b_0, b_1, b_2, \dots$ , we also denote the sequence  $c_0, c_1, c_2, \dots$  as  $a * b$  (so that  $c_n = (a * b)_n$ ).

**Exercise:** Let  $F(x) = 1/(1-x)$  and let  $G(x) = b_0 + b_1x + b_2x^2 + \dots$ . Show that the coefficients of  $F(x)G(x)$  are the partial sums of the sequence  $b_n$ , that is, the coefficient of  $x^n$  in  $F(x)G(x)$  is equal to  $b_0 + b_1 + b_2 + \dots + b_n$ .

**Exercise:** If  $b_n = 2^n$ , see if you can find a formula for the partial sums of the sequence  $b_n$  using generating functions, as opposed to using the formula for the sum of a finite geometric series. (Hint: Express the generating function for the partial sums as a product of rational functions and then use partial fractions.)

Let  $F(x) = 1/(1-x)$ . We'll now study the formal power series  $F(x)^r$ , where  $r$  is a positive integer.

Before reading further, see how much you can discover about  $F(x)^r$  on your own!

First, let's consider  $F(x)^2$ . In this case, the coefficient of  $x^n$  in  $F(x)^2$  is the sum of the first  $n+1$  terms of the constant sequence 1, 1, 1, .... Hence,

$$\left(\frac{1}{1-x}\right)^2 = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

**Exercise:** Verify directly that

$$(1-x)(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots) = 1 + x + x^2 + x^3 + \dots$$

The coefficient of  $x^n$  in  $F(x)^3$  is the sum of the first  $n+1$  coefficients of  $F(x)^2$  coefficients:

$$1 + 2 + 3 + \dots + (n+1) = (n+1)(n+2)/2.$$

**Exercise:** Prove the preceding formula and that it provides the coefficient of  $x^n$  in  $F(x)^3$ .

Thus,

$$\left(\frac{1}{1-x}\right)^3 = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots + \frac{(n+1)(n+2)}{2}x^n + \dots$$

**Exercise:** Verify directly that

$$(1-x)\left(1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots + \frac{(n+1)(n+2)}{2}x^n + \dots\right) = 1 + 2x + 3x^2 + 4x^3 + \dots$$

In fact, the formula for the **binomial series** holds for general exponents and has a useful interpretation for combinatorics:

$$\left(\frac{1}{1-x}\right)^{r+1} = 1 + (r+1)x + \frac{(r+2)(r+1)}{2}x^2 + \dots + C(n+r, n)x^n + \dots$$

**Exercise:** Verify that the general binomial series formula above agrees with the earlier assertions about  $F(x)$ ,  $F(x)^2$ , and  $F(x)^3$ .

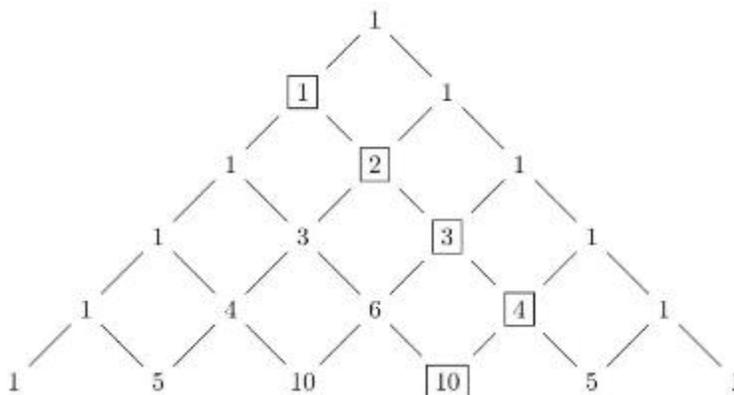
The verification of the general binomial series formula follows from Pascal's identity. If the binomial series formula is true for some value  $r$ , we can deduce that it must be true when you replace  $r$  by  $r + 1$  by looking at the partial sums, which are

$$C(r, 0) + C(r + 1, 1) + C(r + 2, 2) + \dots + C(r + n, n).$$

Pascal's identity tells us that  $C(r + 1 + n, n) = C(r + n, n) + C(r + n, n - 1)$ . It also tells us that  $C(r + n, n - 1) = C(r + n - 1, n - 1) + C(r + n - 1, n - 2)$ . Substituting the last expression for  $C(r + n, n - 1)$  yields  $C(r + 1 + n, n) = C(r + n, n) + C(r + n - 1, n - 1) + C(r + n - 1, n - 2)$ . Again applying Pascal's identity to the last binomial coefficient in the last expression, we find that  $C(r + 1 + n, n) = C(r + n, n) + C(r + n - 1, n - 1) + C(r + n - 2, n - 2) + C(r + n - 2, n - 3)$ . We keep applying Pascal's identity until we arrive at:

$$C(r + 1 + n, n) = C(r + n, n) + C(r + n - 1, n - 1) + C(r + n - 2, n - 2) + \dots + C(r, 0),$$

which shows that the partial sum of interest does turn out to be  $C(r + 1 + n, n)$ . This identity is often referred to as the "hockey stick" rule for Pascal's triangle because the binomial coefficients that appear in it look like a hockey stick inside of Pascal's triangle. The figure below illustrates the case where  $n = 3$  and  $r = 1$ .



**Exercise:** Verify the rule for other hockey sticks in Pascal's triangle. For at least one example, choose any node and show how the hockey stick evolves upward with each use of Pascal's identity. Then state the hockey stick rule for diagonals that start on the right-hand side of Pascal's triangle.

The coefficient of  $x^r$  in  $1/(1 - x)^{r + 1}$  is  $C(r + n, n) = C(r + n, r)$ . We can interpret this coefficient as the number of ways to choose  $r$  objects from a set of  $r + n$  objects. But there's another interpretation in terms of **multisets**. With **multiset counting**, we allow objects to be chosen repeatedly. We claim that  $C(r + n, r)$  is the number of ways to choose  $r$  objects from a set of  $n + 1$  objects when we allow objects to be chosen repeatedly. To see this, note that choosing  $r$  objects in this manner is like putting a total of  $r$  checkmarks on the objects, or, equivalently, placing  $r$  balls (each ball represents a checkmark) into  $n + 1$  different bins (each bin represents an object). We'll prove this interpretation of  $C(r + n, r)$ :

**Interpretation (Multiset counting):** The number of ways to place  $r$  balls into  $n + 1$  distinct boxes is  $C(n + r, r)$ .

To prove the interpretation, we consider a sequence of  $r$  zeros and  $n$  ones. Each placement of balls into boxes corresponds to one of these sequences in the following way: If we visualize the boxes in a row, then we think of each 1 as corresponding to the shared side of two adjacent boxes and the 0's in between two 1's represent balls in a particular box. Adjacent 1's indicate empty boxes, as do leading or ending 1's. (A leading 1 means the first box is empty and an ending 1 means the last box is empty.) The sequences of 0's and 1's are counted by  $C(n+r, r)$ .

**Exercise:** Verify the interpretation for  $n=0$  and  $n=1$ . Write down the sequences of 0's and 1's when  $r=2$  and  $n=2$ .

Let's return to the generating function for the Fibonacci numbers, which will yield our main result:

**Theorem (Binet Formula).** Let  $\phi_+ = \frac{1+\sqrt{5}}{2}$  and  $\phi_- = \frac{1-\sqrt{5}}{2}$ . Then, for all  $n \geq 1$ ,

$$F_n = \frac{1}{\sqrt{5}}(\phi_+^n - \phi_-^n).$$

For what follows, we define  $F(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots + F_{n+1}x^n + \dots$

**Exercise:** Verify that  $F(x) - xF(x) - x^2F(x) = 1$ .

**Exercise:** Verify that  $-\phi_+$  and  $-\phi_-$  are the roots of the quadratic equation  $x^2 + x - 1 = 0$ .

Thus,  $(x^2 + x - 1)F(x) = (x + \phi_+)(x + \phi_-)F(x) = -1$ . Hence,

$$(x + \phi_+)F(x) = \frac{-1}{x + \phi_-} = \frac{1}{-\phi_- (1 - x/(-\phi_-))} = \frac{1}{-\phi_-} \left( 1 - \frac{x}{\phi_-} + \frac{x^2}{\phi_-^2} - \frac{x^3}{\phi_-^3} + \frac{x^4}{\phi_-^4} - \dots \right).$$

If we compare the coefficients for  $x^{n-1}$ , we obtain

$$F_{n-1} + \phi_+ F_n = (-\phi_-)^{-n}.$$

A similar argument gives

$$F_{n-1} + \phi_- F_n = (-\phi_+)^{-n}.$$

If we subtract these equations from each other, we obtain

$$F_n = \frac{1}{\phi_+ - \phi_-} \left( (-\phi_-)^{-n} - (-\phi_+)^{-n} \right).$$

The Binet formula follows once we recognize that  $\phi_+ - \phi_- = \sqrt{5}$  and  $\phi_+ \cdot \phi_- = -1$ .

**Exercise:** The number  $\phi_+$  is the famous **golden ratio**. Calculate  $\phi_+$  and  $\phi_-$  to three decimals each.

**Exercise:** Use the Binet formula to calculate  $F_{20}$ . How small is  $\phi_-^{20}$ ?

The Binet formula allows us to compute Fibonacci numbers directly without the recursive rule. Even better, since the Fibonacci numbers are integers and the contribution from the  $\phi_-$  term is small and only gets smaller with  $n$ , we can obtain  $F_n$  by computing  $\frac{\phi_+^n}{\sqrt{5}}$  and rounding to the nearest integer. Furthermore, to obtain  $F_{n+1}$  from  $F_n$ , we can multiply by  $\phi_+$  and round to the nearest integer.

**Exercise:** Calculate  $F_{15}$  by rounding. Then find  $F_{16}$  and  $F_{17}$  by multiplying by  $\phi_+$  and rounding.

**Exercise:** When can we obtain  $F_{n+k}$  from  $F_n$  by multiplying by  $\phi_+^k$  and rounding? When thinking about this, you might want to consider what you get when you multiply  $F_{10}$  by  $\phi_+^5$  and  $F_5$  by  $\phi_+^{10}$ . (Hint: Simplify  $\phi_+^k \phi_-^n$ .)

**Exercise:** Explain why  $F_n^2$  can be computed by rounding  $\frac{\phi_+^{2n} - 2(-1)^n}{5}$  to the nearest integer. Verify for  $F_8$  and  $F_9$ .

**Exercise:** Calculate  $F_n^2 - F_{n-1}^2$  for  $n = 2, 3, 4$ , and  $5$ . Can you find a nice, general formula?

The idea of rounding with the Binet formula can be used in the following problems from the *Learn by Doing* in Volume 11, Number 3 of this Bulletin (specifically, Problems 11 and 12).

**Useful warm-up:** Prove that  $\phi_+^2 = \phi_+ + 1$ ,  $\phi_+ + 2 = \phi_+ \sqrt{5}$ , and  $\phi_+^4 - 1 = \phi_+^2 \sqrt{5}$ .

**Exercise:** Calculate  $F_n^2 + F_{n-1}^2$  for  $n = 2, 3, 4$ , and  $5$ . Find a general formula that uses rounding.

**Exercise:** Calculate  $F_{n+1}^2 - F_{n-1}^2$  for  $n = 2, 3, 4$ , and  $5$ . Find a general formula that uses rounding.

With good reason, the Fibonacci numbers are a popular topic. For more reading from past issues of this Bulletin, see

- “The Fibonacci Function” from Volume 3, Number 1,
- the Summer Fun problem set “Fibonacci Numbers” from Volume 3, Number 5,
- Anna’s Math Journal from Volume 3, Number 6 and Volume 4, Number 1,
- “A Fibonacci Related Sequence” from Volume 10, Number 4, and
- “Learn by Doing” from Volume 11, Number 3.

For direct generalizations of the lattice path construction of the Catalan and Fibonacci numbers, we highly recommend the book *Counting Lattice Paths Using Fourier Methods* by Shaun Ault and Charles Kicey. Topics include paths for wider intervals, paths that rest (Motzkin paths), trapezoidal grids, and many more.



# The Needell in the Haystack<sup>1</sup>

Supervised Machine Learning: Classification with Candy  
by Anna Ma | edited by Jennifer Sidney

When I was a kid, one of my favorite things to do after a long night of trick-or-treating was to dump my bag of candy into a pile on the floor and one-by-one sort the candy into groups: a pile for Snickers (my favorite), a pile for other chocolate candy, a pile for Smarties, a pile for lollipops, a pile of the non-candy-but-still-good treats, and a pile of the candy I was going to trade away. No one ever sat me down and taught me how to sort my candy but there I was, classifying whether an Almond Joy belonged in the “keep” pile or the “trade” pile (it goes into the trade pile!).

Beyond kids classifying different types of candy, the task of **classification** is integral to our day-to-day functionality as humans. For example, when we look outside and see dark clouds in the sky, we classify the weather as likely to rain so we choose to wear a raincoat. We use verbal cues to classify our parents’ emotions when they talk to us. When we’re looking at food, we classify it as safe to eat or not based on our senses of smell and taste. Whereas humans can use our senses of touch, hearing, sight, smell, and taste to aid in our classification, machines can’t do the same. Instead, computers use data and train algorithms that perform a given classification task.



Figure 1. Halloween candy!

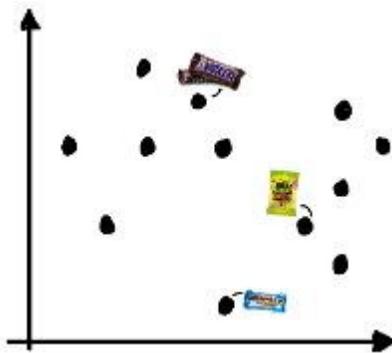


Figure 2. Unlabeled data from (1) plotted on an XY-plane.

In machine learning, classification is heavily relied on for a multitude of tasks. Spam filters classify whether an email is spam or not. Companies classify whether a customer is likely to buy a product, so they can more effectively place advertisements. On phones, facial recognition algorithms try to determine whether a person is the owner of the phone or not. For each of these tasks, data is used to train an algorithm to perform the task. In this edition of “The Needell in the Haystack,” we will begin our dive into the world of classification.

<sup>1</sup> This content is supported in part by a grant from MathWorks. Anna Ma is a Visiting Assistant Professor at the University of California Irvine.

## Data and Labels

Before we can discuss how to perform classification, let's think about what ingredients we need in order to mathematically formulate the task. First, we start with some data. Consider the following set of ordered points

$$\{(2, 3), (5, 1), (8, 2), (7, 3), (8, 4), (9, 5), (1, 5), (3, 5), (3, 7), (5, 5), (8, 6), (4, 6)\}, \quad (1)$$

which we will refer to as our data set. Each data point represents information about a piece of candy: the first number tells us how many colors are on the packaging of the candy, and the second number indicates how sweet the candy is on a scale of 1 to 7. For example, the data point (4, 5) is a Snickers since the packaging has 4 colors and it's quite sweet! As another example, the data point (7, 3) is for Sour Patch Kids; the packaging has 7 colors<sup>2</sup> and is sour and sweet, so it has a lower sweetness score. The data set is plotted in Figure 2. Now that we have data, can we perform classification? Not quite! There's nothing to classify yet. In other words, we need **classes** and **labels** for our data set. Associated labels or classes is what makes classification a **supervised** machine learning approach. We are supervising learning by telling our algorithm what we want it to learn: how to label data points based on our desired labels. There is also an entire suite of **unsupervised** machine learning techniques where the algorithm is allowed to learn without labeled data, but that will be explored in a future edition of "The Needell in the Haystack."

Our labels will determine what type of task our algorithm learns: Do we want it to learn to identify candy that has chocolate or not? Whether or not the candy has nuts? Or is there something else we want our algorithm to learn? For this example, we will train an algorithm to identify whether or not a candy will be put into the "keep" or "trade" pile:

$$\{\text{keep, trade, trade, trade, trade, trade, trade, keep, keep, keep, keep, trade, keep}\}. \quad (2)$$

Figure 3 shows the data from (1) with colors and shapes to indicate the labels given in (2): the blue circles are candies we want to keep, and the red stars are candy we want to trade. Now that we have labels, can you draw a line or curve to separate the blue circles from the red stars in Figure 3?

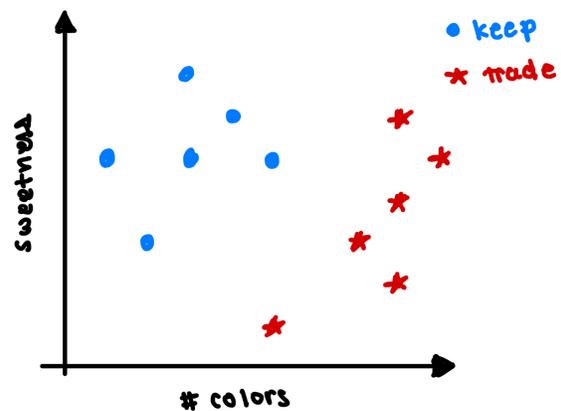


Figure 3. Labeled data using labels (2) plotted on an XY-plane.

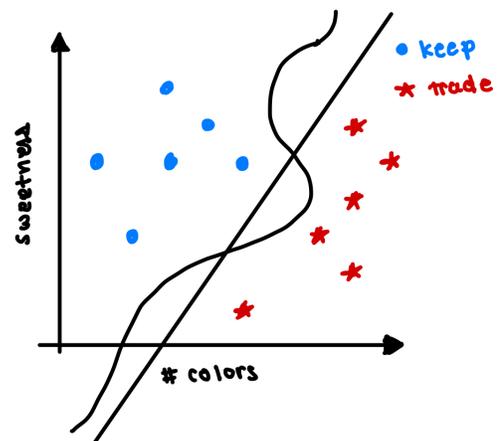


Figure 4. There are many possible ways to separate the blue circles from the red stars. Two possible ways to do it are shown: using a line, and using a curve.

<sup>2</sup> Numbers of colors of packaging of candy may vary.

## Linear Classification

The data shown in Figure 3 can actually be separated in many ways! Some examples are shown in Figure 4. To simplify things, we will separate the classes using a line, or a **linear classifier**. For the two-dimensional case, a linear classifier can be defined as a line:

$$y = mx + b, \quad (3)$$

where  $m$  and  $b$  are unknown parameters and  $x$  and  $y$  come from the data set. Once we have obtained a line, we can then classify the data depending on which side of the line the data is on: if the data is to the right of the line, we will label it as “trade,” and if it appears on or to the left of the line, we will label it as “keep,” as shown in Figure 5. More precisely, we can define our linear classifier as follows: Given  $m$  and  $b$  in Equation (3), our candy classifier  $f(x, y)$  outputs

$$f(x, y) = \begin{cases} \text{"keep"} & \text{if } mx + b \leq y \\ \text{"trade"} & \text{if } mx + b > y, \end{cases} \quad (4)$$

where  $x$  and  $y$  are real numbers.

In addition to classifying our original data set, we can classify new data points using (4). Given that a candy has  $x = 3$  colors on its wrapper and a sweetness level of  $y = 2$ , would we keep or trade this candy? By comparing  $3m + b$  with 2, we can determine whether it's a traded or kept candy. Based on (4), if  $3m + b \leq 2$ , we should keep the candy; otherwise, we should trade the candy.

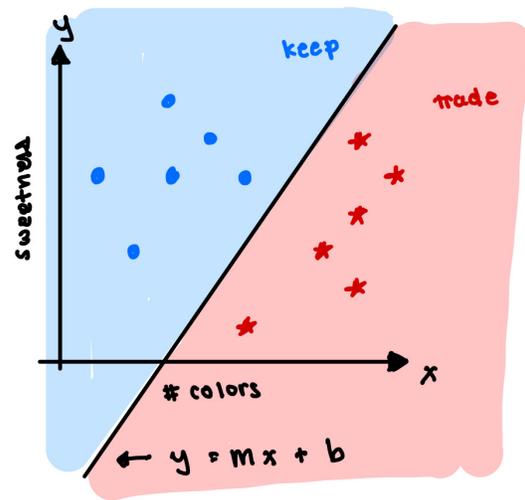


Figure 5. A linear classifier splits the  $XY$ -plane into two regions. One region indicates the points that represent candy that will be kept, and the other region represents candy that will be traded away.

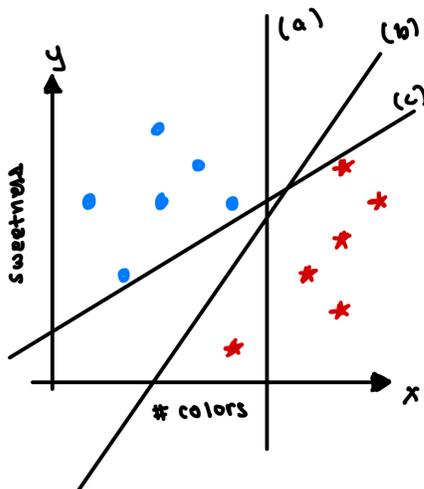


Figure 6. Which linear classifier is the “best” classifier?

## Evaluating Classifiers

There isn't just one line that is able to separate the data; in other words, there isn't a *unique* classifier. However, some classifiers will be better for certain purposes than others. Consider Figure 6. Notice that all the lines, labeled (a)-(c), separate the data points into two sides. Which do you think is the “best” classifier of the three? What makes it the best?

In order to compare different linear classifiers, we need to come up with ways to measure how good a line is as a classifier. One option is to count the number of misclassified points in the data set. An example of a misclassified point is a point that would be labeled as “keep” by the classifier even though its true label is “trade.” In

other words, if the classifier  $f(x_i, y_i)$  does not output the correct label, then we add 1 to the total number of misclassified points. Thus, the fewer misclassified points, the better the classifier. Using the number of misclassified points, we can evaluate the classifiers shown in Figure 6.

In Figure 7, we can see that for classifier (a), there is one point that is misclassified; for classifiers (b) and (c), there are no misclassified points. Since a smaller number of misclassifications indicates a better classifier, we can conclude that with respect to the number of misclassifications, (b) and (c) are better than (a). However, how do we distinguish which is better, (b) or (c)? Do you notice any interesting differences between these two classifiers in Figure 7?

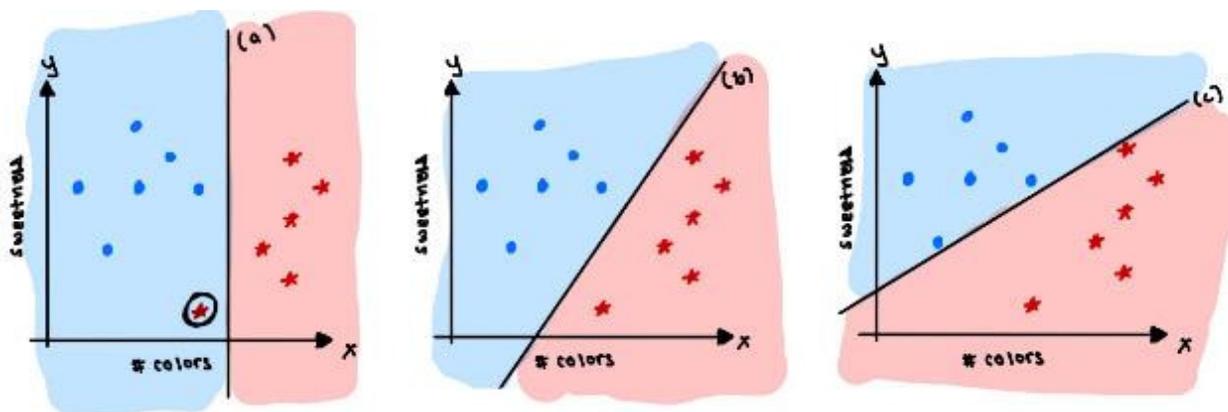


Figure 7. Three possible linear classifiers for the data set (1) and corresponding labels (2).

Figure 8 includes a new, unlabeled data point (2, 2.9). Note that this data point or candy is very, very similar to the candy data point at (2, 3); it has the same number of colors on the packaging but is just the slightest bit less sweet. So which category do you expect this new candy to be in, trade or keep? Depending on assumptions we make about the problem, we can argue that the candy belongs to either class. For example, we might argue that because we already kept a candy that is similar to the new candy (i.e., we already kept the candy (2, 3)), we may want to trade this new candy away. On the other hand, we can also argue that we may want to keep this new candy (2, 2.9), because it's similar to candy we know we like (I always keep the Snickers variations, like the almond Snickers). Now, looking at Figure 8, note that this data point would be labeled differently depending upon whether we use line (b) or line (c). Using line (b), the classifier would label the data point as keep, whereas using line (c), the classifier would label the data point as trade.

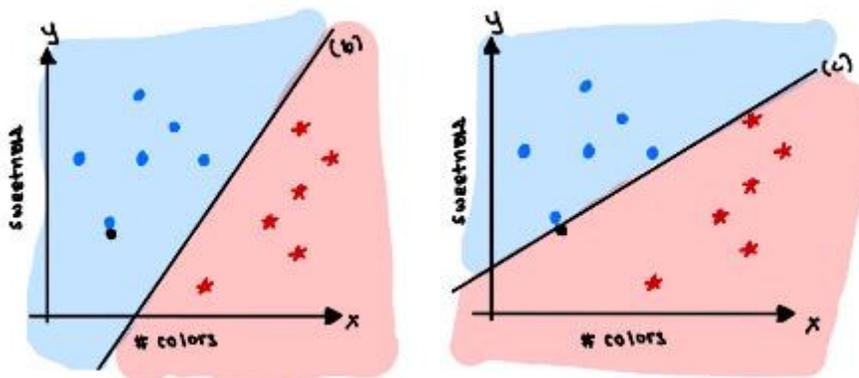


Figure 8. A new data point (unlabeled in black) is added to the data set.

In most real-world settings, it's advantageous to use linear separators that are farther from the data, as in classifier (b) in Figure 8. This is because these classifiers are less prone to a problem in machine learning known as **overfitting**. Overfitting occurs when our machine learning model becomes too sensitive to small changes in the data set. This sensitivity can be seen when a small change in the data causes an algorithm – or in our case, a linear classifier – to return a completely different output. Because real-world data is typically noisy, we want to avoid overfitting so that our algorithm doesn't have drastically different outputs when new, noisy data comes our way.

### Try It Yourself!

Let's suppose we want a linear classifier that accomplishes two things: (1) it separates two classes and (2) it's as far away as possible from the data points. Try coming up with your own separators in Figure 9. Can you find the slope  $m$  and intercept  $b$  for your own linear separators? Are there data points that make finding a good linear classifier difficult? Which data set is easier to separate? Stay tuned for the next edition of "The Needle in the Haystack" to learn how we can algorithmically solve for the linear classifier!

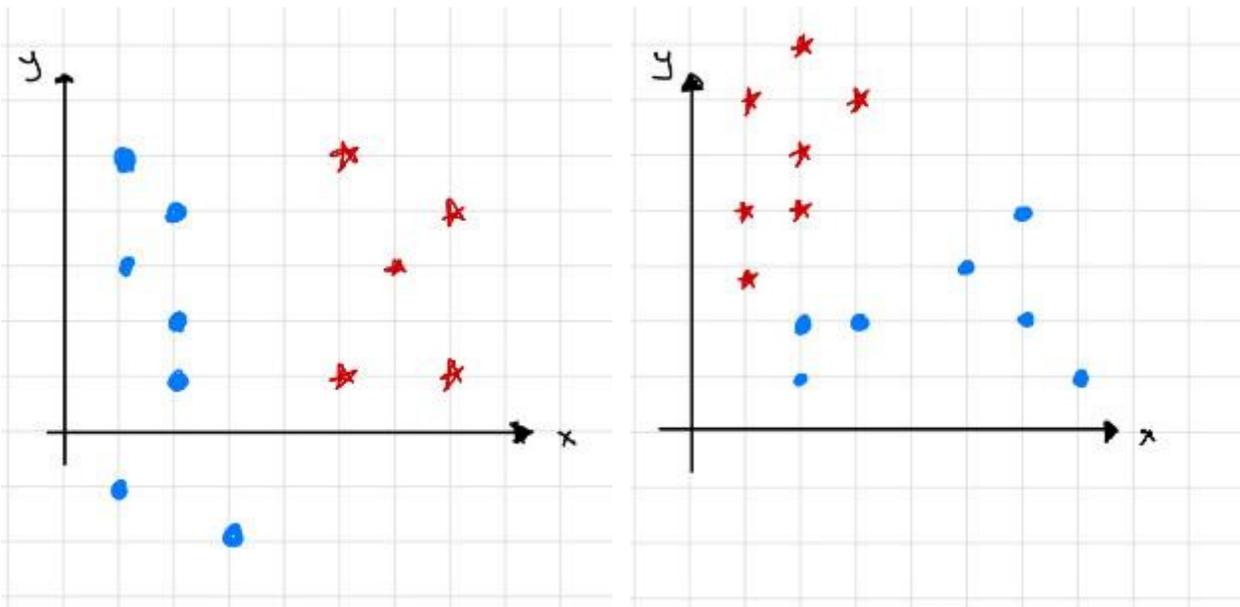


Figure 9. Try it yourself: For each of the data sets, determine the linear classifier that both separates the two classes and is as far away as possible from the data points.

# Notation

by Girls' Angle Staff  
edited by Jennifer Sidney

Good notation facilitates communication. To learn notation, use it. Practice makes perfect!

In this “Notation Station,” let’s talk about the general challenge of gaining facility with new notation, a frequent and familiar situation in mathematics!

First of all, why do mathematicians seem to love to create notation? After all, mathematics is about ideas, not symbols. The answer is that mathematicians do love discussing ideas – so much so that when they communicate, stating all the ideas would produce an avalanche of words that would require a major effort to parse. Therefore, they introduce notation to encapsulate ideas. In theory, then, notation is a tool to help clarify and organize all of the beautiful mathematical thoughts.

When you encounter notation for the first time, however, it can easily feel as if the notation does the opposite, ostensibly obscuring the math in a cloud of mysterious symbols.

To get a handle on notation, let’s think about a notation that people are taught at such a young age that most people cannot even recall the struggle they went through to assimilate it: decimal numbers. If you’ve written a check to pay for something, then you know that you must write the dollar amount for the check both as a decimal number and using words. In one space, you write “\$128.43.” But in another, much larger space, you have to write:

*“One hundred twenty-eight dollars and forty-three cents.”*

(In fact, many people find it too cumbersome to write even that, so compromise by only writing out the dollar amount in words: *“One hundred twenty-eight and 43/100.”*)

In your opinion, which way of expressing the value of the check is easier to grasp?

If you feel that “\$128.43” is easier, that is because at some point earlier in your life, you put in a lot of effort to learn this decimal notation for expressing quantity. You might have forgotten just how much effort you put into this task. As a reminder, the full task usually begins with just understanding the notation for the numbers 0 through 9 – the single-digit numbers. We often fill pages of notebooks with those symbols, just as we once filled pages of notebooks with dozens of copies of each of the letters of the alphabet, both lower- and uppercase. We then progress to multi-digit numbers, typically going in stages from 2-digit to 3-digit numbers. At that stage, people often regard working with numbers that require more than 3 digits as “harder,” and the decimal point has not even been introduced yet. When the decimal point is finally introduced, a lot of effort is put into translating between numbers with decimal points and fractions. Does this jog your memory of those days?

If you find the word version of the number easier to grasp, it simply means that you have yet to put in the effort described in the previous paragraph.

With all of that repetitive effort, the mind begins to associate the symbol directly with the concept. We reach a point where immediately upon seeing “24,” we understand – seemingly without any translation – that this is the number of hours in a day.

As we become more adept at using decimal numbers, we note things like “24 is more than 15,” “0.25 is the same as 1/4,” “1000 is 10 cubed,” “1,234 is even,” etc., while hardly thinking at all.

To develop this facility, these efforts are unavoidable.

They’re unavoidable because notation is an arbitrary choice. For example, in Chinese, the symbols for the digits 1 through 9 look entirely different from the Arabic ones that we’ve adopted. Gaining facility with notation is, ultimately, a memorization task.

The key to memorization is repetition. Force yourself to use the notation, but don’t rush. Go slowly enough that you are able to use the notation correctly. This is an occasion where the old adage “haste makes waste” is particularly apt.

So much of learning math is psychological, and the ease with which you pick up new notation does depend on your attitude. Go in with the attitude that it’s all fun, and it will be. Treat it like it’s a major drag, and it will become a painful process. Perhaps if it were up to you, you would have used a different-looking notation, but there’s no use complaining. It is what it is, and for the sake of communication, we must accept that. Just remind yourself that notation is all arbitrary anyway. So if words are written left to right and top to bottom in all of the books that are printed, well, that’s just the way we’ll have to write, too (if we wish to participate in society, at least).

Is it worth the effort?

Well, would you rather write/read

“In the year 2020, the population of Boston was 689,326.”

or

“In the year two thousand twenty, the population of Boston was six hundred eighty-nine thousand three hundred twenty-six.”

?

Before your efforts,

$$2 \int_{-1}^1 \sqrt{1-s^2} ds$$

might look like a piece of abstract art. But after your efforts, it’ll look like  $\pi$ , the area of the unit circle.

Before your efforts, symbols may look like alien hieroglyphics. After your efforts, you’ll be seeing ideas.

# Summer Fun!

In the previous issue, we presented the 2022 Summer Fun problem sets.

In this issue, we give solutions to many of the problems. Our solutions may be terse and, in some cases, are more of a hint than a solution. We prefer not to give detailed solutions before we know that most of the members have spent time thinking about the problems. The reason is that *doing* mathematics is very important if you want to learn mathematics well. If you haven't tried to solve these problems yourself, you won't gain as much when you read these solutions.

If you haven't thought about the problems, we urge you to do so *before* reading the solutions. Even if you cannot solve a problem, you will benefit from trying. By working on the problem, you will force yourself to think about the associated ideas. You will gain familiarity with the related concepts and that will make it easier and more meaningful to read other's solutions.

With mathematics, don't be passive! Be active!

Move your pencil and move your mind – you might discover something new.

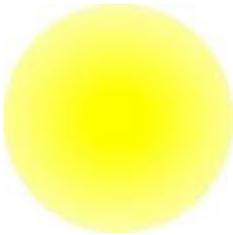
Also, the solutions presented are *not* definitive. Try to improve them or find different solutions.

Solutions that are especially terse will be indicated in **red**. Please do not get frustrated if you read a solution and have difficulty understanding it. If you run into difficulties, we are here to help! Just ask!

*Members: Don't forget that you are more than welcome to email us with your questions and solutions!*

Please refer to the previous issue for the problems.

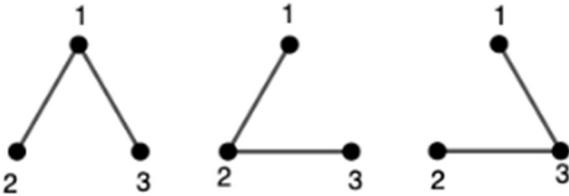
# Summer Fun!



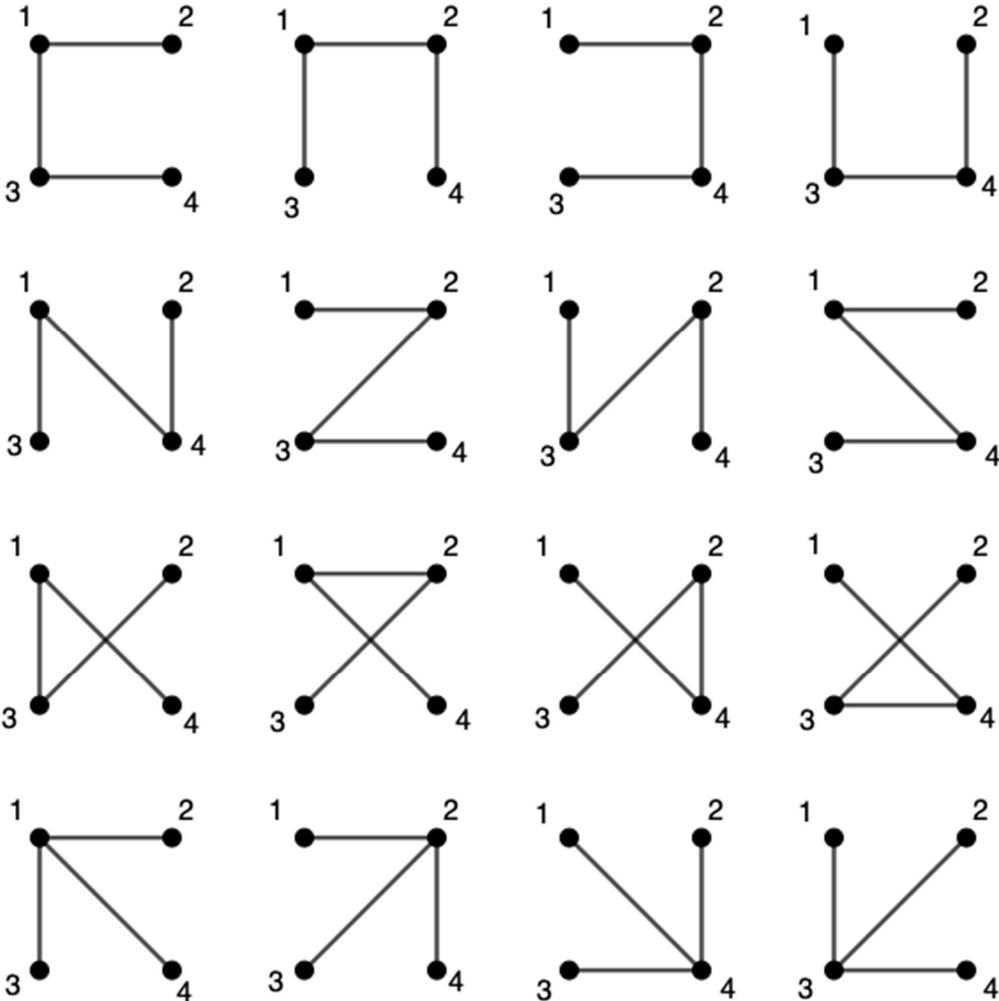
# Counting Trees and Parking Cars

by Laura Pierson

1. The 3 possible labeled trees on 3 vertices are:



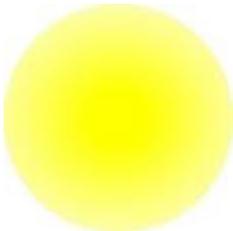
2. The 16 possible labeled trees on 4 vertices are:



3. See the cover!

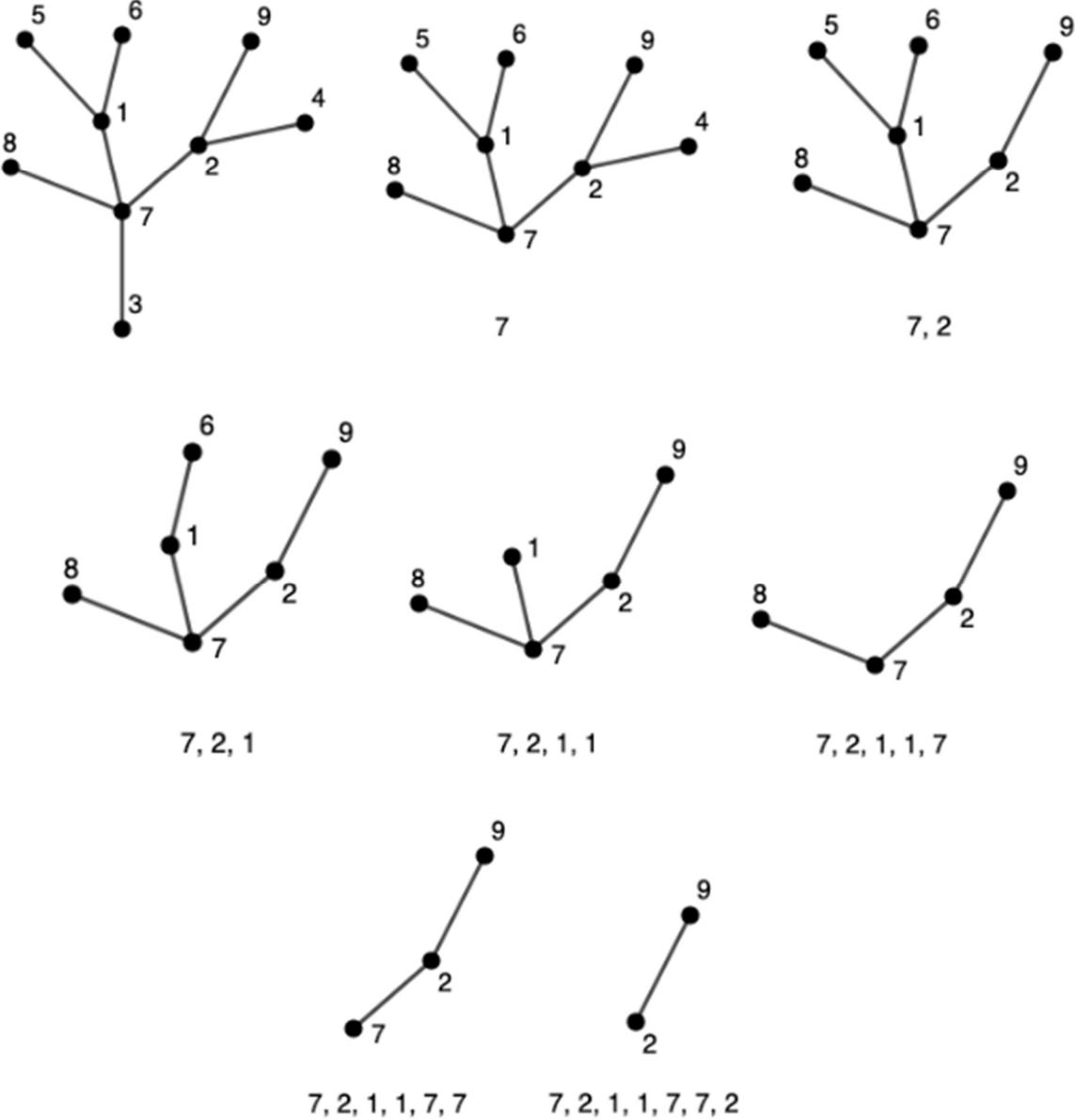
4. If you're stuck, read on!

# Summer Fun!



5. A Prüfer sequence has length  $n - 2$ , and there are  $n$  choices for each term, so the total number of possible sequences is  $n^{n-2}$ .

6. We follow the sequence of steps below to construct the Prüfer sequence, like in the example:



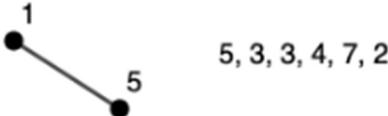
The final sequence is 7, 2, 1, 1, 7, 7, 2.

# Summer Fun!



7. We can reconstruct the tree from the sequence 5, 3, 3, 4, 7, 2 as follows.

- We know that at the start of constructing the Prüfer sequence from the tree, 5 was the neighbor of the smallest leaf. Since 1 doesn't appear anywhere in the sequence, it must have started as a leaf, so 1 was this first smallest leaf. So, we connect 1 to 5. Now we will consider just the part of the sequence after the 5, since the 5 has been dealt with.



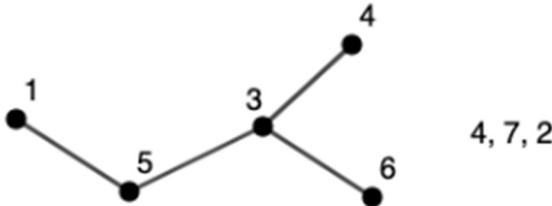
- After removing vertex 1, the new smallest leaf must have been connected to vertex 3. Since 5 no longer appears in the sequence but 2, 3, and 4 do, the new smallest leaf at this point must have been 5. So there is an edge from 5 to 3. We can now draw this edge and remove 3 from the sequence.



- After 1 and 5 were removed, the smallest remaining leaf must have been connected to vertex 3. Since 2, 3, and 4 still show up in the sequence, they must not have been leaves yet, and 1 and 5 were already removed, so the smallest remaining leaf at this point must have been 6. So we add an edge from 3 to 6.

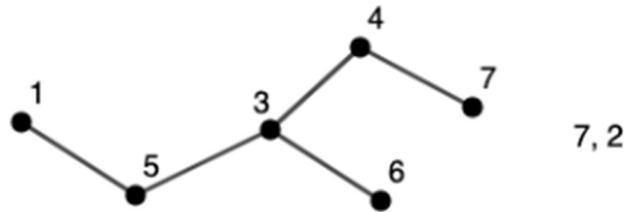


- The next smallest leaf must have been connected to vertex 4. Vertex 1 was already removed, and vertex 2 still shows up in the sequence, so it can't have been a leaf yet. However, 3 no longer shows up in the sequence, so it was a leaf at this point. So 3 is connected to 4.

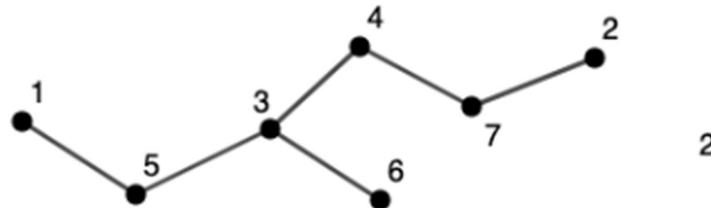


- The next smallest remaining leaf should be connected to vertex 7. We've already used 1 and 3, and 2 isn't a leaf yet since it shows up in the sequence. However, 4 no longer shows up in the sequence, so it would have been the smallest remaining leaf at this step. So we connect 4 to 7.

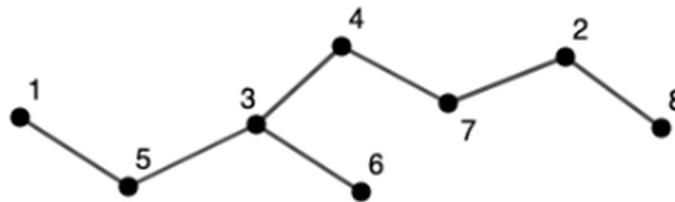
# Summer Fun!



- At this point, vertices 5, 1, 6, 3, and 4 would have been removed, and the smallest remaining leaf should be connected to vertex 2. The smallest remaining leaf must have been 7 at this step, so we draw an edge from 7 to 2.



- Finally, we need to add vertex 8. Since 2 is the only vertex that was never a leaf, vertex 8 needs to be connected to vertex 2. This completes our tree!



8. There is a unique Prüfer sequence for each tree, and Problem 7 shows how we can go in the other direction to find a unique tree for each Prüfer sequence. This shows that there is a one-to-one correspondence between Prüfer sequences and trees. Since the number of Prüfer sequences is  $n^{n-2}$  by Problem 5, the number of trees is also  $n^{n-2}$ .

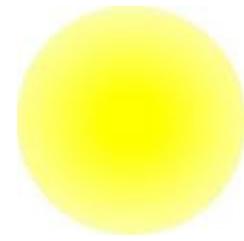
9. The 3 parking functions for 2 cars are (1, 1), (1, 2), and (2, 1).

10. The 16 parking functions for 3 cars are:

(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 1),  
 (1, 2, 2), (1, 2, 3), (1, 3, 1), (1, 3, 2),  
 (2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 2, 1),  
 (2, 3, 1), (3, 1, 1), (3, 1, 2), (3, 2, 1).

11. A. There are  $n$  cars, and for each car there are  $n + 1$  possible preferred parking spaces, so  $n + 1$  options per car. Thus, the total number of preference functions is  $(n + 1)^n$ .

# Summer Fun!



B. We can turn a preference function where space  $i$  is left open into one where space  $i + 1$  is left open by increasing the preferred spot of each car by 1, including “wrapping around” so  $(n + 1) + 1$  becomes 1, i.e., working modulo  $n + 1$ . We can also go in the other direction by decreasing the preferred spot of each car by 1 (again, wrapping around modulo  $n + 1$ ). This shows that the number of preference functions where space  $i$  gets left open is the same for every  $i$ .

C. If we have a parking function, then spots  $1, 2, \dots, n$  get taken before any car ever reaches spot  $n + 1$ , so that must be the spot left open.

For the other direction, we will show that if the sequence of preferences is not a parking function, then spot  $n + 1$  gets taken, and thus is not the spot left open. One possibility for the sequence to not be a parking function is if some car prefers spot  $n + 1$ . If there is such a car, then either that car takes spot  $n + 1$ , or it cannot because a car before it took that spot. Either way, the spot gets taken.

The other possibility for a sequence that is not a parking function is that all the cars prefer a spot between 1 and  $n$ , but at some point, a car finds that its preferred spot  $k$  is taken, and so are all the spots  $k + 1, k + 2, \dots, n$ . In that case, this car will just keep driving and park in spot  $n + 1$ , so again, spot  $n + 1$  gets taken.

D. From part C, the number of parking functions equals the number of possible preference functions where parking space  $n + 1$  is the one left empty. From part B, each of the  $n + 1$  parking spaces is equally likely to be left empty, so spot  $n + 1$  should be the one left empty  $1/(n + 1)$  of the time. By part A, the total number of preference functions is  $(n + 1)^n$ . Putting all this together, the number of preference functions where spot  $n + 1$  is left empty is

$$\frac{1}{n + 1} (n + 1)^n = (n + 1)^{n-1}.$$

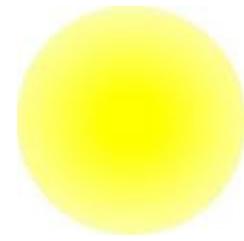
12. A. We construct a Dyck path from a tree as follows. Draw the tree like a family tree, with the root 0 at the top, and the other vertices coming down from it. A vertex which is directly below another vertex is called is “child” (like in a family tree!).

We will go through the nodes in “depth first search” order. This means that for each node, if it has any children, we visit its smallest child next. Otherwise, we go back up until we reach a level with a child we haven’t visited yet, and visit that child next. In the given example, we visit the nodes in the order

0, 3, 1, 5, 6, 11, 10, 2, 7, 9, 4, 8.

Now, for each node we visit, we add a sequence of up steps to the Dyck path, whose labels correspond to its children from biggest to smallest. Then we add a single down step.

# Summer Fun!



In this case, we proceed as follows:

- We start at the root, vertex 0. Its children are 3, 6, and 10, so we add up steps labeled 3, 6, and 10, followed by a down step.
- Next, we visit vertex 3. Its children are 1 and 5, so we add up steps labeled 1 and 5, followed by a down step.
- We visit vertex 1 and see that it has no children, so we just add a down step.
- We visit vertex 5, which also has no children, so again we add a down step.
- We go back up and visit vertex 6. Its one child is vertex 11, so we add an up step labeled 11, followed by a down step.
- We go to vertex 11, which has no children, so we just add a down step.
- We go back up to vertex 10. We add up steps for its children 2, 4, and 8, followed by a down step.
- We visit vertex 2, and add up steps for its children 7, and 9, followed by a down step.
- Vertex 7 has no children, so we just add a down step.
- Vertex 9 has no children, so again, we just add a down step.
- We go back up to vertex 4, which has no children, so we just add a down step.
- We visit vertex 8, which again has no children, so we add a final down step.

B. As suggested in the hint, let's list the cars who prefer space 1, then space 2, and so on:

- Cars 3, 6, and 10 prefer space 1, and are the children of vertex 0.
- Cars 1 and 5 prefer space 2.
- No cars prefer space 3.
- No cars prefer space 4.
- Car 11 prefers space 5.
- No cars prefer space 6.
- Cars 2, 4, and 8 prefer space 7.
- Cars 7 and 9 prefer space 8.
- No cars prefer space 9.
- No cars prefer space 10.
- No cars prefer space 11.

We can now see that the cars who prefer space  $i$  correspond to the up steps along the  $i$ th upward-going diagonal of the Dyck path (which may be empty if there are multiple down steps in a row).

C. Part A gives a one-to-one correspondence between trees and labeled Dyck paths, so the number of trees equals the number of labeled Dyck paths. Part B gives a correspondence between labeled Dyck paths and parking functions, so the number of labeled Dyck paths equals the number of parking functions. Thus, the number of trees equals the number of parking functions!

# Summer Fun!



# Making OJ – The Hard Way!

by Girls' Angle Staff

5. You can mix any rational concentration of orange juice and serve it up in any amount that is a multiple of the greatest common divisor of  $x$  and  $y$ . (A rational concentration occurs when the ratio of the amount of pure orange juice (i.e., squeezed right out of an orange) to the amount of water added is a rational number. A rational number is any number that can be expressed as a ratio of two integers.)

To see this, suppose we have a rational number expressed as  $p/q$  where  $p$  and  $q$  are positive integers. We can put  $px$  amount of pure orange juice and  $qx$  amounts of water into one of the large cups to create  $px + qx$  amount of liquid at the concentration  $p/q$ . We can repeat this process until we effectively have an unlimited supply of orange juice at the concentration  $p/q$ .

We can then proceed to produce an amount of this mixture equal to the greatest common divisor of  $x$  and  $y$ . To see this, let's focus on the remainders that multiples of  $x$  leave when you divide those numbers by  $y$ . The first few multiples of  $x$  are  $x, 2x, 3x, 4x, \dots$ . The first of these multiples to also be a multiple of  $y$  is the least common multiple of  $x$  and  $y$ , which is equal to  $xy/d$ , where  $d$  is the greatest common divisor of  $x$  and  $y$ . The first  $y/d$  multiples of  $x$  all leave *different* remainders when you divide them by  $y$  (why?). Also, all such remainders are multiples of  $d$  (why?), and there are exactly  $y/d$  such remainders. Hence, some multiple of  $x$ , say  $kx$ , leaves a remainder of  $d$  when you divide it by  $y$ . In other words, there exists a nonnegative integer  $m$  such that  $kx - my = d$ . We can then fill up a second large cup  $k$  times using the  $x$  liter cup, then empty  $my$  liters from this by using the  $y$  liter cup to obtain  $d$  liters of  $p/q$ -concentrated OJ!

(Why are no other concentrations and amounts possible?)

6. We can add and subtract water in the amount of 1 or  $\sqrt{2}$  liters. By performing these multiple times, we can obtain  $a + b\sqrt{2}$  liters of water for any integers  $a$  and  $b$  such that  $a + b\sqrt{2}$  isn't negative. So, the question is about finding integer values of  $a$  and  $b$  so that  $a + b\sqrt{2}$  is within  $1/100$  of  $1/2$ . This is equivalent of finding a multiple of  $\sqrt{2}$  that is within  $1/100$  of a so-called **half-integer**, which is a number obtained by dividing an odd number by 2, i.e.,  $1/2, 3/2, 5/2$ , etc.

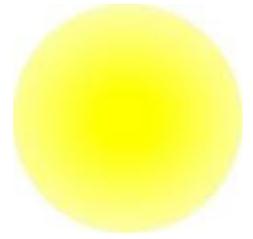
We could compute multiples of  $\sqrt{2}$  until we finally find one that is within  $1/100$  of a half-integer. (The first such multiple is  $35\sqrt{2}$ .) But is there a less computationally intensive way?

Suppose  $a + b\sqrt{2}$  is very close to  $1/2$ , that is, suppose  $a + b\sqrt{2} = 1/2 + x$ , where  $x$  is small in absolute value. We can rearrange this to  $\sqrt{2} = (1 - 2a)/(2b) + x/b$ . In other words,  $(1 - 2a)/(2b)$  is a good rational approximation to  $\sqrt{2}$ . Finding rational approximations to irrational numbers is a well-studied topic. We urge you to think about it more, and if you get stuck, see, for example, David Speyer's articles *Approximating Real Numbers by Fractions* in Volume 4, Numbers 2 and 3 of this Bulletin.

# Summer Fun!

# Fermi Questions

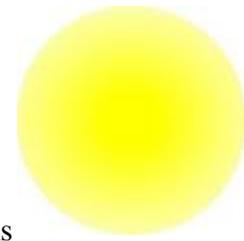
by AnaMaria Perez and Josh Josephy-Zack



Because Fermi questions are not meant to be determined exactly, for each problem, we estimate an answer and then provide the base 10 logarithm of the answer rounded to the nearest whole number.

1. 4. In fact, there are exactly 10,000 circles in the cover art, then, if you count the circles in the text, there are an additional 7 more.
2. 9. There are about 15,000 Starbucks locations and about 50,000 people attended the Tokyo Olympics.
3. -10. The average American female lifespan is about 80 years, which is about  $2.5 \times 10^9$  seconds, and an eye blink is about  $1/3$  of a second.
4. 1. There's about 1000 feet of wool inside a baseball and the Green Monster at Fenway Park is about 37 feet high.
5. 7. A standard Rubik's Cube is about 6 centimeters on a side and an Olympic-sized swimming pool is approximately 25 meters by 50 meters by 3 meters.
6. -15. A 747 expends about  $1.4 \times 10^8$  watt per hour and it takes about 7 hours to fly from London to New York. (Recall that an answer in yotta-watt hours was called for.)
7. 6.
8. -9. Usain Bolt's top speed is about 44,000 meters per hour and a light year is about  $10^{16}$  meters.
9. 17. There are about 46 million acres of farmland in Kansas and the head of a pin is about  $10^{-6}$  square meters. (There are about 4,000 square meters in an acre.)
10. 13. There are about a billion cases of the common cold in the United States each year and about 35,000 letters in the Constitution of the United States.
11. 10. Krispy Kreme produces about 22,000 donuts every day, and each donut is about 6 centimeters in height. The distance to the Moon is about 384 million meters.
12. 10. About a billion servings of coke are produces every day in the United States and each one has roughly 40 milligrams of caffeine on average.

# Summer Fun!



13. -35. The output power of the sun is about  $3.8 \times 10^{26}$  watts and the amount of energy in a fully charge iPhone 13 battery is about 400 joules.
14. 19. The total mass of all the oxygen in Earth's atmosphere is about  $5 \times 10^{21}$  grams and the volume of a Magic 8 Ball is about 800 cubic centimeters.
15. 11. The world's tallest tree is about 100 meters high and the length of a carbon dioxide molecule is about  $3 \times 10^{-10}$  meters.
16. 8. About  $5 \times 10^8$  copies of Harry Potter have been sold and each is about 3 centimeters thick. A worm is about 35 centimeters long.
17. 12. The average annual rainfall in the Amazon rainforest is about  $10^{16}$  liters. An Amazon delivery van can hold about 7,000 liters.
18. -102. A titanium atom has a mass of about  $8 \times 10^{-26}$  kilograms and the mass of the planet Neptune is about  $10^{26}$  kilograms.
19. 2. The equator of mercury is about 15,000 kilometers long and you can draw a line about 50 kilometers long with a pencil.
20. 16. Assuming current power consumption, New York City will use about  $4 \times 10^{18}$  calories in a century. A twinkie offers about 260 calories of energy.

Bonus 1. 3. The author's favorite numbers are 73 and 22.

Bonus 2. For both A and B, we can solve for the expected value of the area of the triangle by using integration. Let  $(r, a)$  and  $(R, A)$  be the polar coordinates of the two randomly picked points. Then the coordinates of the vertices of the triangles are  $(0, 0)$ ,  $(r \cos a, r \sin a)$ , and  $(R \cos A, R \sin A)$ , and its area is  $|rR \sin(A - a)|/2$ .

A. The probability that a point fall inside a little rectangular region in polar coordinates of dimensions  $dr$  by  $dA$  is equal to  $r dr dA / \pi$ . Therefore, the expected value of the area is:

$$\int_0^{2\pi} \int_0^1 \int_0^{2\pi} \int_0^1 \left| \frac{rR \sin(A-a)}{2} \right| \frac{rR}{\pi^2} drdadRdA = \frac{1}{18\pi^2} \int_0^{2\pi} \int_0^{2\pi} |\sin(A-a)| dadA = \frac{4(2\pi)}{18\pi^2} = \frac{4}{9\pi}.$$

B. The probability that a point fall inside a little rectangular region in polar coordinates of dimensions  $dr$  by  $dA$  is equal to  $dr dA / (2\pi)$ . Therefore, the expected value of the area is:

$$\int_0^{2\pi} \int_0^1 \int_0^{2\pi} \int_0^1 \left| \frac{rR \sin(A-a)}{2} \right| \frac{1}{(2\pi)^2} drdadRdA = \frac{1}{32\pi^2} \int_0^{2\pi} \int_0^{2\pi} |\sin(A-a)| dadA = \frac{4(2\pi)}{32\pi^2} = \frac{1}{4\pi}.$$

# Summer Fun!

# Calendar

Session 31 (tentative schedule): (all dates in 2022)

September	8	Start of the thirty-first session!
	15	
	22	
	29	
October	6	
	13	
	20	
	27	
November	3	
	10	
	17	
	24	Thanksgiving - No meet
December	1	

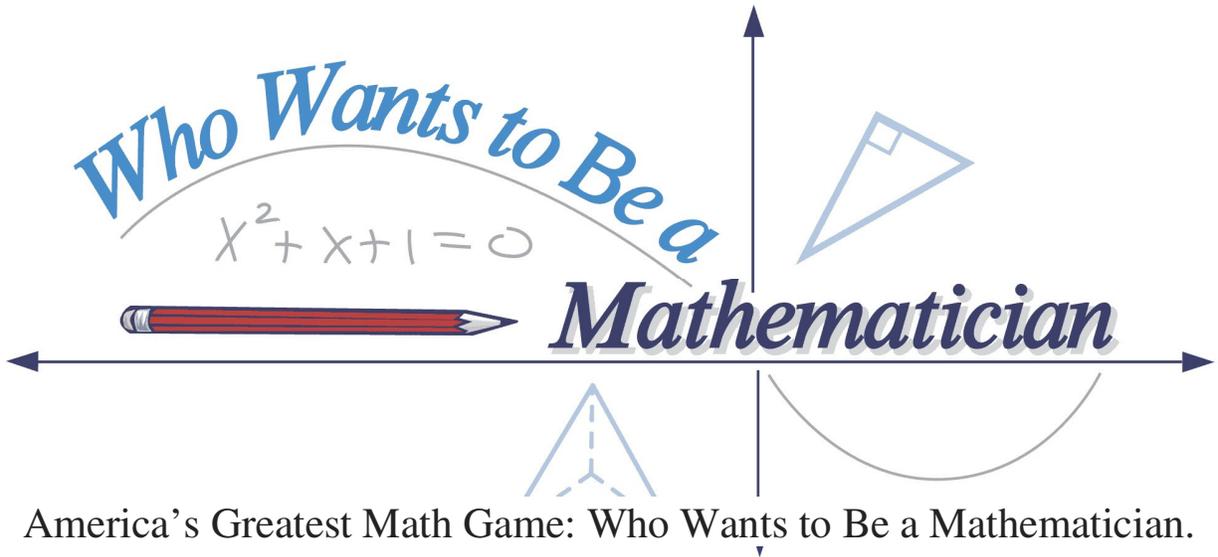
Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at [girlsangle@gmail.com](mailto:girlsangle@gmail.com). For more information and testimonials, please visit [www.girlsangle.org/page/math\\_collaborations.html](http://www.girlsangle.org/page/math_collaborations.html).

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: [girlsangle@gmail.com](mailto:girlsangle@gmail.com).

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America's Greatest Math Game: Who Wants to Be a Mathematician.

(advertisement)

# Girls' Angle: A Math Club for Girls

## Membership Application

**Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.**

Applicant's Name: (last) \_\_\_\_\_ (first) \_\_\_\_\_

Parents/Guardians: \_\_\_\_\_

Address (the Bulletin will be sent to this address):

Email:

Home Phone: \_\_\_\_\_ Cell Phone: \_\_\_\_\_

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

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The \$50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- Enclosed is a check for \$50 for a 1-year Girls' Angle Membership.
- I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to [girlsangle@gmail.com](mailto:girlsangle@gmail.com).



**A Math Club for Girls**

# Girls' Angle Club Enrollment

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

**Who are the Girls' Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls' Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

**What is Community Outreach?** Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

**How do I enroll?** You can enroll by filling out and returning the Club Enrollment form.

**How do I pay?** The cost is \$20/meet for members and \$30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

**Where is Girls' Angle located?** Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at [www.girlsangle.org/page/calendar.html](http://www.girlsangle.org/page/calendar.html) or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory  
Yaim Cooper, Institute for Advanced Study  
Julia Elisenda Grigsby, professor of mathematics, Boston College  
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign  
Grace Lyo, assistant dean and director teaching & learning, Stanford University  
Lauren McGough, postdoctoral fellow, University of Chicago  
Mia Minnes, SEW assistant professor of mathematics, UC San Diego  
Beth O'Sullivan, co-founder of Science Club for Girls.  
Elissa Ozanne, associate professor, University of Utah School of Medicine  
Kathy Paur, Kiva Systems  
Bjorn Poonen, professor of mathematics, MIT  
Liz Simon, graduate student, MIT  
Gigliola Staffilani, professor of mathematics, MIT  
Bianca Viray, associate professor, University of Washington  
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin  
Lauren Williams, professor of mathematics, Harvard University

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

# Girls' Angle: Club Enrollment Form

Applicant's Name: (last) \_\_\_\_\_ (first) \_\_\_\_\_

Parents/Guardians: \_\_\_\_\_

Address: \_\_\_\_\_ Zip Code: \_\_\_\_\_

Home Phone: \_\_\_\_\_ Cell Phone: \_\_\_\_\_ Email: \_\_\_\_\_

Please fill out the information in this box.

**Emergency contact name and number:** \_\_\_\_\_

**Pick Up Info:** For safety reasons, only the following people will be allowed to pick up your daughter. Names:

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

**Eligibility:** Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

**Personal Statement (optional, but strongly encouraged!):** We encourage the participant to fill out the optional personal statement on the next page.

**Permission:** I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

\_\_\_\_\_  
(Parent/Guardian Signature) Date: \_\_\_\_\_

Participant Signature: \_\_\_\_\_

Members: Please choose one.

- Enclosed is \$216 for one session (12 meets)
- I will pay on a per meet basis at \$20/meet.

Nonmembers: Please choose one.

- I will pay on a per meet basis at \$30/meet.
- I'm including \$50 to become a member, and I have selected an item from the left.

I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to [girlsangle@gmail.com](mailto:girlsangle@gmail.com). Also, please sign and return the Liability Waiver or bring it with you to the first meet.

**Personal Statement (optional, but strongly encouraged!):** This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

**Girls' Angle: A Math Club for Girls  
Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

\_\_\_\_\_ ,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: \_\_\_\_\_ Date: \_\_\_\_\_

Print name of applicant/parent: \_\_\_\_\_

Print name(s) of child(ren) in program: \_\_\_\_\_