Girls' Angle Bulletin

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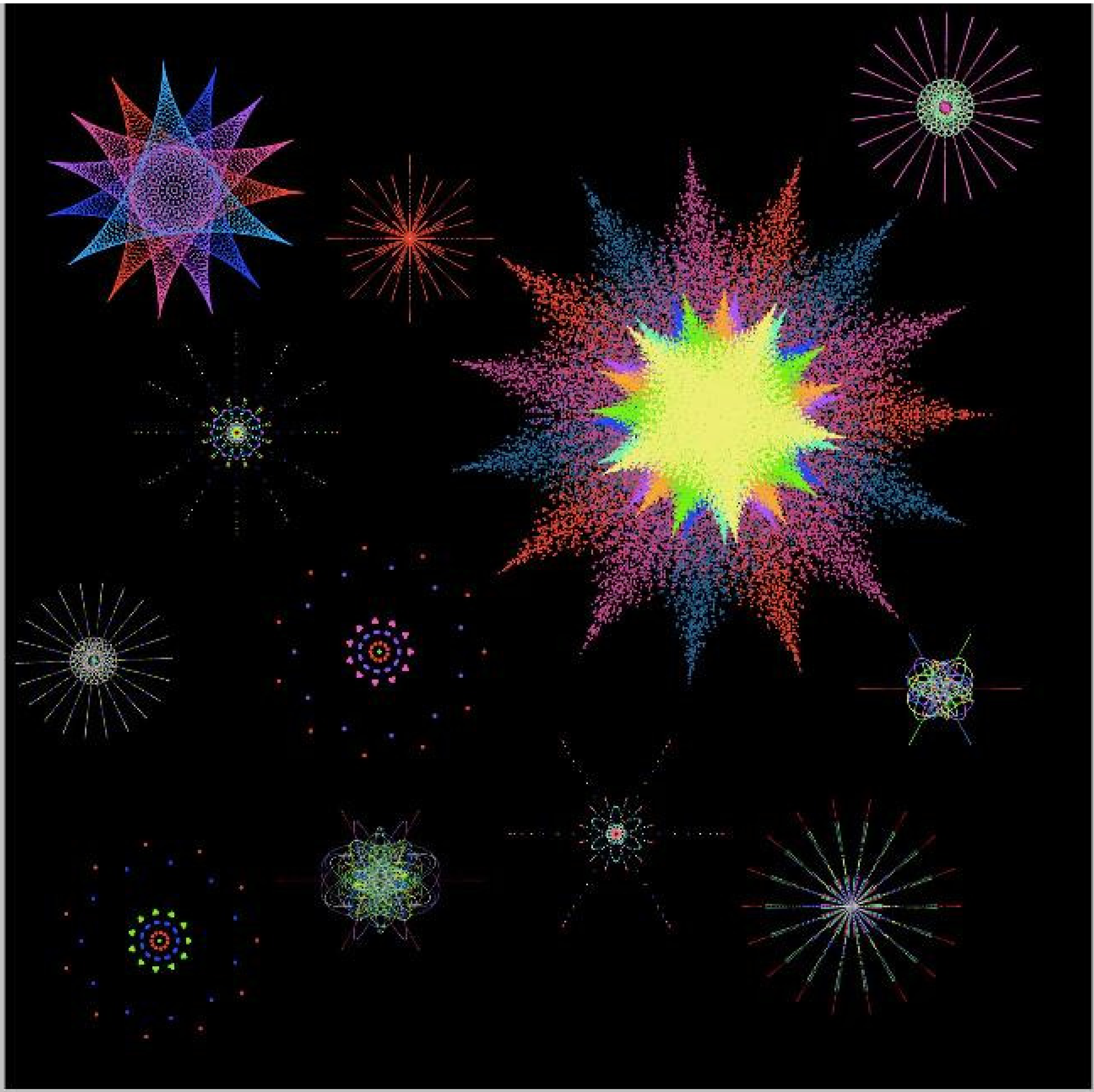
To Foster and Nurture Girls' Interest in Mathematics



From the Founder

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Girls' Angle Bulletin

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Girls' Angle: A Math Club for Girls

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On the cover: *Gaussian Fireworks* by Ellen Eischen. For details, see page 16.

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An Interview with Tullia Dymarz, Part 1

Tullia Dymarz is Associate Professor of Mathematics at the University of Wisconsin. She earned her doctoral degree in mathematics from the University of Chicago under the supervision of Benson Farb. She was awarded an NSF Career Grant in 2016. In addition to her research and teaching, she runs a program for high school girls called Girls Math Night Out!

Ken: I wanted to start by urging our readers to read the wonderful companion essay by Isa Barth on page 10, as I'll assume knowledge of that content and takeoff from there.

Isa quoted you as saying that in college you had a friend who you took many classes with, but that he was, "Never interested in the beauty of math." So, the first question I have for you is, "Could you please give us an example of mathematical beauty?"

Tullia: Okay. This friend of mine and I actually also went to high school together, and in 12th grade, we had a math class that was for those of us who had finished calculus and the rest of the high school curriculum. The teacher showed us the perfect shuffle where you take your deck of cards, and you split it in half, and then you interleave the cards perfectly. Actually, there are two perfect shuffles, depending on whether the top card stays on top, which is called an "outer shuffle," or the top card becomes the second card from the top, which is called an "inner shuffle." Our teacher had us do perfect outer shuffles over and over, and then after eight times, he had us check the deck again, and we saw it was back to where it started!

You really have to be thinking about infinite objects for quasi-isometry to be interesting.

I thought this was just like magic, of course, back then. But then, we learned the math behind it when we started talking about permutations, iterating permutations, and how you could look at it abstractly and determine exactly how many times you need to perform the same perfect shuffle before a deck of any length would come back to its original order.

I really enjoyed the abstract theory that came out of card shuffling, whereas, my friend, would have been like, "Great. Eight times," and been ready to move on. Okay, maybe I'm exaggerating a little bit, but I thought that was something practical that led to a lot of abstract beauty.

Ken: That's a great example. Can you pinpoint what makes it beautiful? Or is that something of a mystery?

Tullia: Yes, right. Because, of course, beauty is in the eye of the beholder, and – I'm not sure. I think for me, it was the way the card trick was the starting off point for the general, abstract theory. I found that I was less interested in the application than the abstract understanding, or theory, that it led to.

Later on in my undergraduate years, I learned about groups and how the card shuffle was really just one little aspect of this abstract notion of groups. And so, I really liked how you could take some concrete thing, then leave it and go off into this universe that exists, really, only in your

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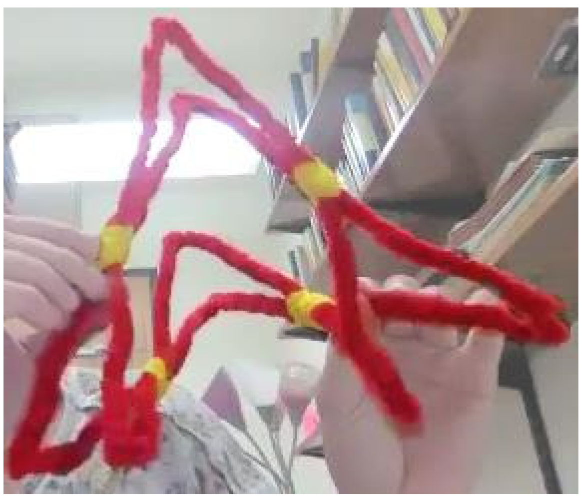
head. So, yes, I think it was the abstraction that drew me and what I found beautiful.

Ken: Is this what motivates you to do mathematics – a search for mathematical beauty?

Tullia: Yes. Right now, I think it's like an exploration of these sort of objects that only exist in the abstract and in my head. That's not to say that I don't try to make concrete realizations. I'm actually in my office right now where you can see[1] that I'm very much into working with pipe cleaners to build models of things. But really, it's exploring these abstract spaces that others have defined, or that I've defined, and trying to understand their properties.



Ken: Wow, what is that pipe cleaner contraption a model of?

Tullia: This is called a Diestel-Leader graph. It's actually a very small portion of it. You have to imagine it goes off to infinity in both directions. It's part of an infinite graph or network.

Ken: It's not at all clear to me how that would continue.

Tullia: So, every yellow node has two edges going up and two edges going down. Now imagine the same at the top where the edges going up from two of the yellow nodes meet. They actually meet in another yellow node, although this particular model does not have the top and bottom nodes colored yellow yet. But at the top where they meet, there would be another yellow node with two edges going up from it, and there's already two going down. This continues to infinity in both directions.

This represents one of the objects that I'm very interested in studying. Of course, I can't make an infinite model, but I can make small finite chunks of it, and try to help myself visualize what the whole thing looks like.

Ken: Would you describe one of your own favorite discoveries and say something about how you discovered it?

Tullia: Actually, it was about these objects. But maybe a better way explain it is to first talk about metric spaces. A metric space is a space that has a notion of distance. For example, on this model the distance between one yellow node and another can be computed by finding a path between them that traverses the fewest number of pipe cleaners. The distance is then the number of pipe cleaners traversed. In a network, that's often how we define the distance between nodes of the network.

My favorite result is actually not that hard to state. I showed that there are no bijections between two of these kinds of metric spaces that doesn't distort distance too much. That is, there's no way to match up the points of one with those of the other that doesn't distort distance too much.

[1] Our interview was conducted over Zoom.

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Let me give you an example. If you just take the positive integers 1, 2, 3, 4, etc. then compare them to the positive even integers 2, 4, 6, 8, etc., by dividing by two, you get a matching from the positive even integers to the positive integers and you've only distorted the distance by a factor of two. For example, the points 2 and 4 are a distance 2 apart, and they get mapped to 1 and 2, which are a distance 1 apart, so they get squished together by a factor of 2.

So I studied this sort of thing but on the more complicated space of these infinite graphs, of which these pipe cleaners model a small portion.

The way I discovered this was by deciding that on these Diestel-Leader graphs, you can't do anything like this trick of dividing by 2. It's a basic observation, but somehow, nobody had thought of it before.

Ken: I saw that you've written a lot about a concept called "quasi-isometry." Is what you're talking about now related to quasi-isometry?

Tullia: Yes, exactly.

Ken: So, what is quasi-isometry?

Tullia: Well, let me start with something that people are more used to, although they might not use the same term in school, and that's the notion of isometry. Isometry is something that preserves distances exactly. So, for example, if you have two circles of the same size drawn on a piece of paper, well, you can kind of slide one to exactly overlap the other one. Sliding doesn't change distance at all. That's an isometry.

In school they're called "equivalent circles," and you can try the same thing with squares, or triangles, or any shape, really. And in school, you study conditions for when two shapes are equivalent, such as two triangles are equivalent if they have the same side lengths. If two triangles have the same side lengths, then you can slide, flip, and rotate one triangle onto the other so that they overlap exactly.

We say those two triangles are "equivalent," but really, we should maybe say that they are "isometrically equivalent," because we've matched one to the other, all without changing distances at all. But then, you could relax that, and you do in school, too, when you're talking about similar triangles, where a similar triangle is one that has the same angles, but maybe not the same edge-lengths. So, you're allowed to scale one triangle to the other.

When you scale, think about what's happening to distance. You're stretching or shrinking distances by some factor, and you can do this not just with triangles, but any other sort of shape. So, for example, any two circles are similar because you can scale one to the other. Any two squares are also similar.

But a square and a rectangle that's not a square, are not similar, because to make them look alike you might have to stretch one pair of sides but shrink the other pair. But maybe you want to even consider shapes where you stretch in one direction and shrink in another to also be equivalent for certain purposes, and so this gives another notion of equivalence that is getting close to the notion of quasi-isometry.

Maybe I say, "Okay, I'll allow stretching different parts differently, but at most by a factor of ten and at least by a factor of two," or something like that. And then you can say, "Well, the square can be equivalent to

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For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Tullia Dymarz and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

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Thank you and best wishes,
Ken Fan
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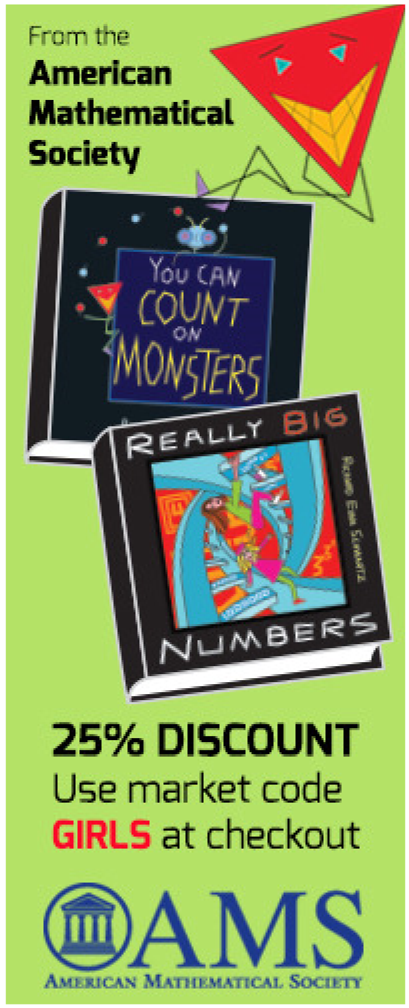

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Tullia Dymarz: On Connection in Math and in Life[1]

by Isa Barth | edited by Jennifer Sidney Silva

About the Author: Isa Barth is an undergraduate mathematics student at the University of Wisconsin-Madison. Currently in their second year, they humbly acknowledge that their mathematics education is still, in some sense, in its early stages – although they are especially interested in mathematical logic and topology. For now, though, they are enjoying exploring everything the field has to offer (honoring a certain professor's advice to "take as many interesting classes as possible"). They look forward to a future in which they can work in research and discover even more about math and our world.

Tullia Dymarz claims that she never made a proper decision to study mathematics; for her, it was "always the obvious thing to do." Perhaps this is because mathematics has always surrounded her: raised in Edmonton as the daughter of a physicist, she was encouraged by her father to explore mathematics growing up. He gave her private lessons, and while she admits that she was never particularly fond of her father's teaching style, his mentorship fostered a deep appreciation for mathematics in her from early on.

After breezing through her high school's mathematics curriculum with a year to spare before graduation, she was presented with a unique opportunity. In the company of a small handful of peers, she took a special course curated by her 12th grade teacher: a sampling of college-level math concepts, from linear algebra to basic group theory. She was especially fond of one lesson, in which methods of shuffling cards were analyzed using permutations. "It wasn't until later that I learned the more abstract math behind the concepts," she reflects, "but there were so many really cool projects at the time." The class showcased many of the unusual ways mathematics dictates everyday life, and gave many reasons for Dymarz to further fall in love with it.

Dymarz stayed close to home for her undergraduate studies, opting into the honors mathematics program at the University of Alberta at the start of her first year. This was common for many students at the time, and there were several draws to going to college in the area where you grew up. "If you go to [university] where you went to high school, you keep those friendships through college," she explains, "especially if they're interested in what you're interested in."

Dymarz does cherish the many connections she has made with her peers over the years; indeed, she has always been able to find a place among her classmates. However, even when bonding with others interested in math, she found through high school and her undergraduate years that few of them appreciated math in quite the same way she did. "I had one high school friend, we were always competing.… We took some of the same honors math classes at university together, but he was never interested in the beauty of math. He wanted to do something practical." Even in her honors program, she was the only one in the small group of students with her sights set on graduate school.

It should come as no surprise that she *did* manage to find others who appreciated the "beauty of math" in her graduate program at the University of Chicago. And this was not the only change graduate school presented: in a class size of twenty-four students, she was one of eight girls – a far cry from the male-dominant ratios she was so familiar with.

As she and her peers tackled countless "impossible" problems together, she found herself relying more on her fellow females than ever before. Yet this was not merely due to the rigor of the curriculum; there was also a change in the gender dynamic she was unable to ignore. "If you had

[1] A version of this essay was submitted to the AWM/MfA Essay Contest in 2021.

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two guys talking, most often they would talk math. But when a guy talked to a girl, oftentimes the conversation drifted away from math," she admits. "You had to be more forceful with them. And if there were fewer women there, I believe it would have been easier to give up." With the increasing need for collaboration that graduate school demanded, the support of her female peers proved to be integral, and they created an environment in which they could motivate and challenge each other.

Perhaps the greatest connection Dymarz feels she has been able to make was with her advisor. In fact, in some ways she cared more about her advisor's personality than their specialization. She opted to develop her thesis outside of her original area of interest, instead choosing a mentor who could match her abundance of enthusiastic energy. For Dymarz this wasn't a sacrifice; in her eyes all mathematics is interesting, thus fascinating research prospects abound.

She ended up selecting geometric group theory as her main area of study. When she's in a bar and is asked what she does for work, she likes to explain her specialty – somewhat jokingly – in terms of the fundamentals. "I tell them I study triangles," she chuckles, before eventually going on to draw several comparisons to things like social networks and distances travelled in a car. Her description, ultimately, proves itself fitting – and it brings to light many instances in which pure mathematics intersects with reality.

As a professor at the University of Wisconsin-Madison, Dymarz continues her "study of triangles" and to foster a love of mathematics in others. On top of her duties as a professor and researcher, she runs the university's *Girls Math Night Out!* program, which presents a unique opportunity to high school girls interested in engaging in STEM research. According to Dymarz, the main goal of the program is to provide teens with an alternative environment to math meets or other typical mathematics programs, which tend to be purely competitive. It offers a chance for teens to experience the same female-centric environment Dymarz experienced in graduate school, one which emphasizes discovery and camaraderie.

While Dymarz does laugh that she "keeps a good work-life balance" and has never felt particularly overwhelmed by mathematics, she remains mindful of the ways math surrounds us and connects to our world in unexpected ways. Indeed, one of her favorite emails from a student details a rather unorthodox case of this: while teaching a proof-based course at Chicago through her graduate program, she received a note from a student in the humanities, who claimed that learning proofs had made her "completely upend" the way she wrote essays. Dymarz well understands the student's sentiment. "It creates a unique way of thinking," she notes about proofs, "an abstract way of thinking. I think that's the power of it, this system of thinking about arguments… it can be really useful, no matter what you do."

That email details one of many stories that show the breadth of mathematics' reach – and the depth of Dymarz's impact on her own students. As someone helping to nurture the next generation of mathematicians, she remains confident that opportunities for success in the field are plentiful. After all, she has seen firsthand many of the ways that math stays relevant – through the sheer abundance of connections to our world, as well as the ways it connects people to one another.

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Mathematical Buffet: Creativity Counts

Curated by Ellen Eischen

Like many art forms, mathematics can be strikingly beautiful. In 2021, the Jordan Schnitzer Museum of Art in Eugene, Oregon hosted *Creativity Counts*, an exhibit that Professor Ellen Eischen organized to share a creative side of mathematics with the broader community.[1] In the next few pages, you can get a taste of the exhibit, which features mathematical artwork produced by students and faculty at the University of Oregon. To explore further, check out pages.uoregon.edu/eeischen/CreativityCounts/.

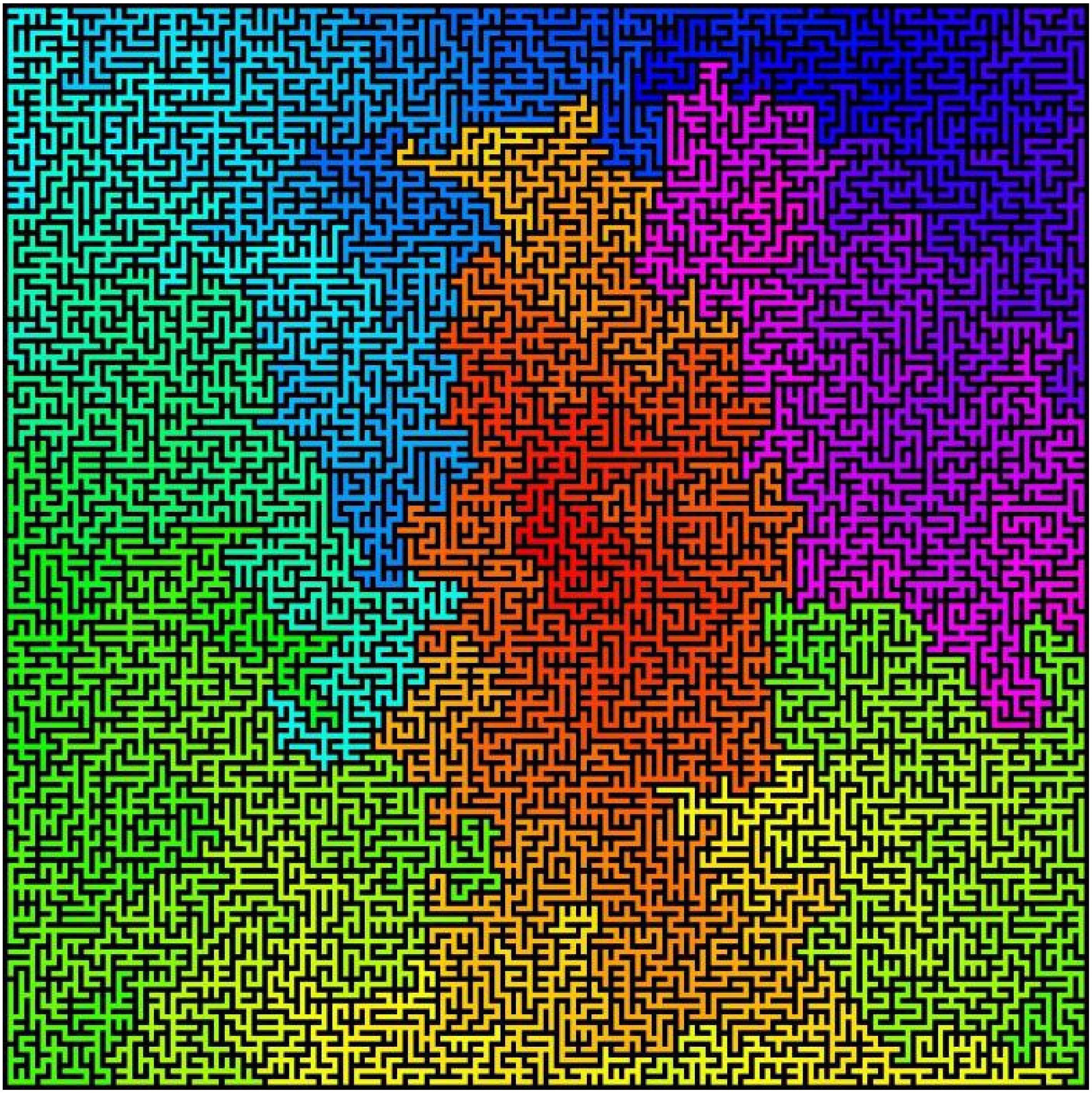

Figure 1. Minotaur's Paradise, by Cruz Godar

Ever wanted to make your own maze? If you've tried, you might find it's not as easy as you might think. Draw just one long path, and it's too easy; draw lots of small ones and you'll often run into the same problem. If even a small maze is tricky to make, a massive one seems downright impossible -- and yet pick up any puzzle book for children and you'll see pages upon pages of them. How do they do it? Enter Wilson's Algorithm. It's a method that makes maze-making as easy as following a recipe, and what's more, it produces a truly random one— every single possible maze of a fixed size has the same chance of being drawn.

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[1] Many thanks are due to consultant Heather Barnes from Improv@Work, whose workshops helped the artists featured in *Creativity Counts* strengthen their skills to communicate effectively about mathematics with people of all mathematical backgrounds. This project was supported by NSF CAREER grant DMS-1751281 and the Williams Fund.

Figure 2. Engineered for Success, by Cruz Godar

Have you ever looked at a fern up close? The tiny leaves are exactly the same shape as the big ones. This is an example of something called a fractal, and while it may seem firmly an artifact of the physical world, it's just as naturally a product of math. Placing a tiny green dot on a canvas, repeatedly applying a simple kind of function called an affine transformation to it, and drawing a new dot in every place it visits, we wind up with the Barnsley fern, named after its creator. The function we apply may have been carefully engineered, but it's no less remarkable that something so beautiful and complex can arise from something so simple.

You can watch the fern being generated and also play with a variety of fractals on Cruz Godar's website at cruzgodar.com/applets/applets.html.

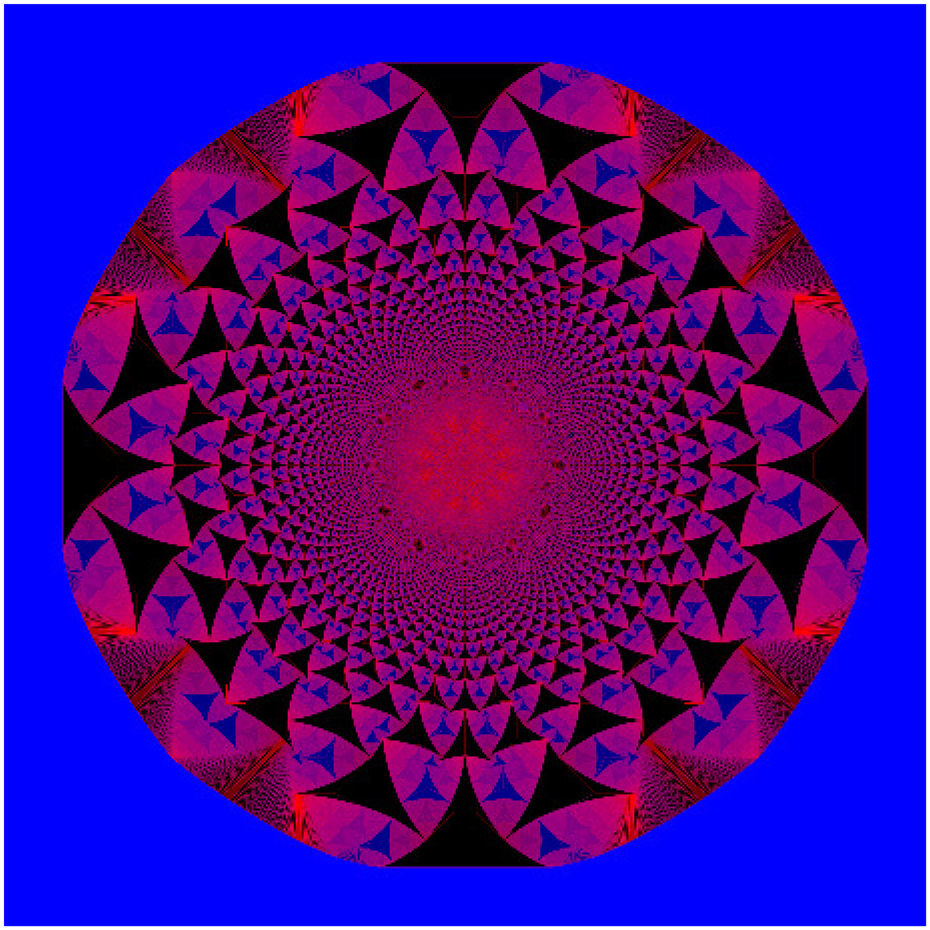


Figure 3. The Sand Reckoner, by Andy Huchala

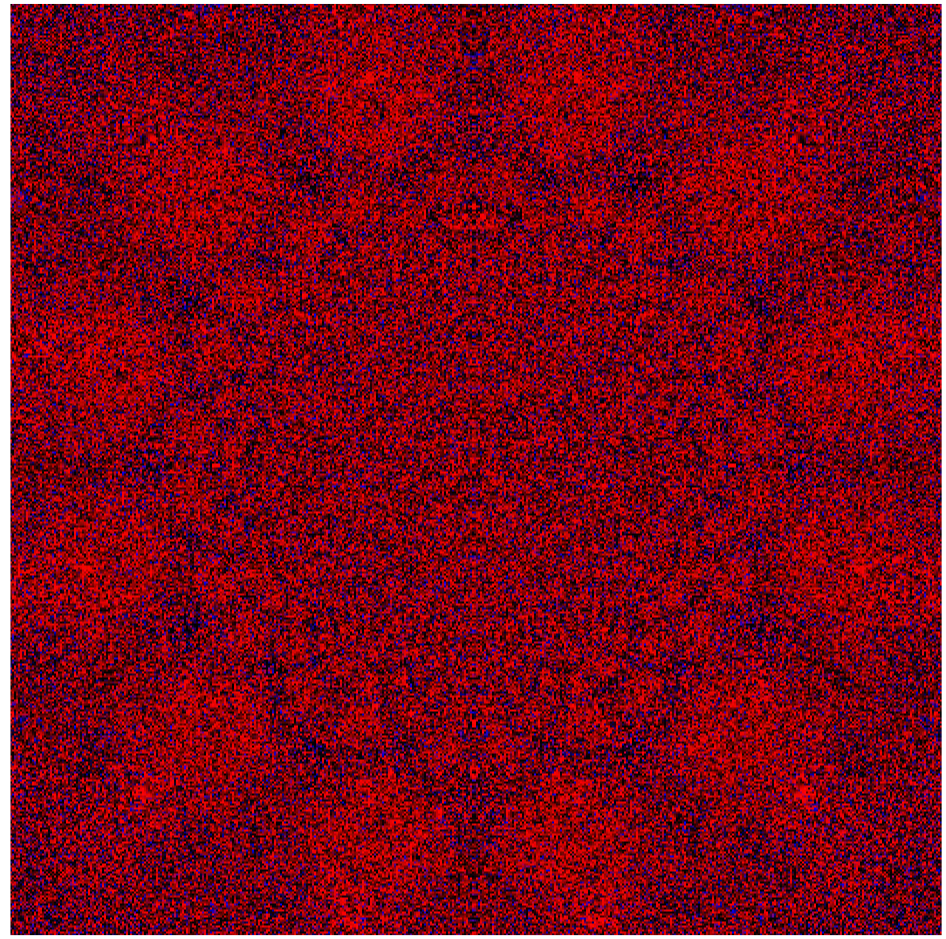
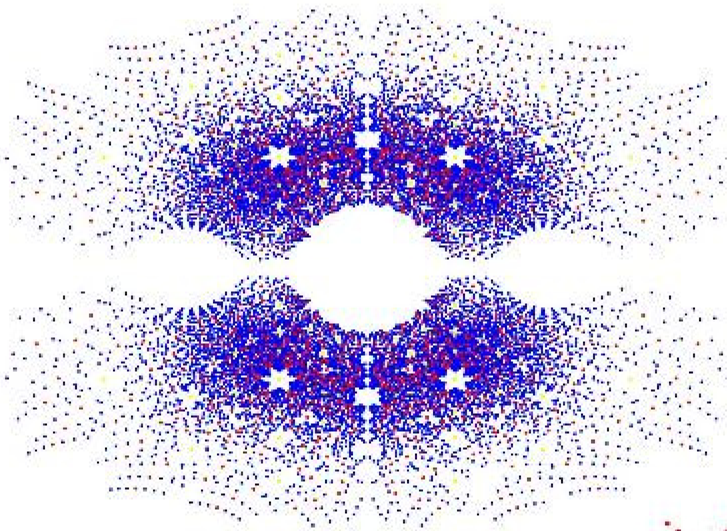
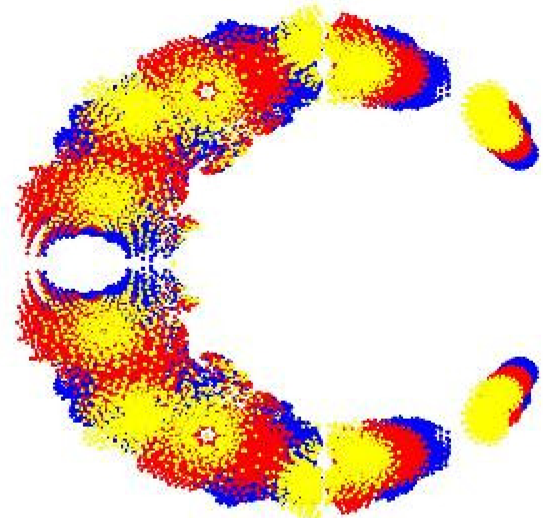


Figure 4. Zooming in on the center of the image.

This piece attempts to capture the spirit of the infinite and print it on a single page. It uses what's called an *Abelian Sandpile model*, which is a fractal that colors "grains of sand" by their slope. A fractal is a kind of mathematical picture with a repeated motif no matter how far in or out you zoom. The Abelian Sandpile model achieves this by stacking a large number of grains of sand in the center of a grid, and then "topples" it onto the adjacent 4 tiles. The taller the pile of sand at the start, the more times you can topple, and the closer to the illusion of infinity you get. Adding color depicting the different heights of sand helps us visualize the resulting fractal structure.





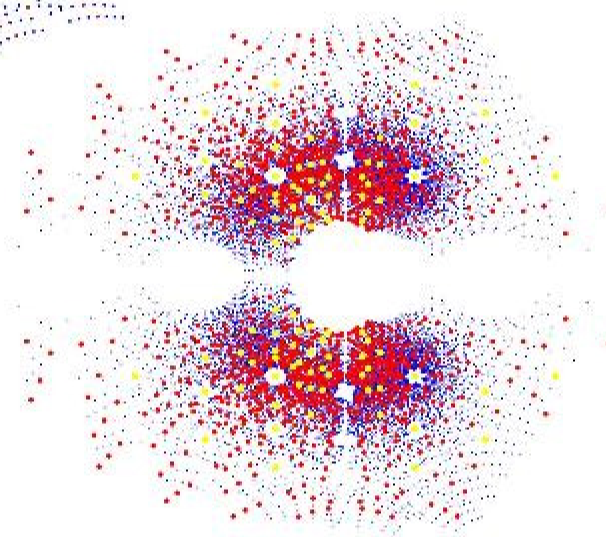


Figure 5. Constellations of Mathematics I, by Azusena Rosales Suares.

Figure 6. Constellations of Mathematics II, by Gabby Bennett.

Figure 7. Constellations of Mathematics III, by Chloe Miller.

Have you ever looked up at the stars in a city? The bright lights in the city make it harder to see all the stars, but you know they are out there. No matter where we stand and look up at the sky at night, we cannot see all the stars, even though we can sometimes see more and more of them filling the sky. Likewise, when we plot certain numbers called *roots of families of polynomials*, we see interesting patterns, but there are always infinitely many more numbers that arise in this way. The stars have patterns called *constellations*. Here, we see constellation-like patterns in our plots.

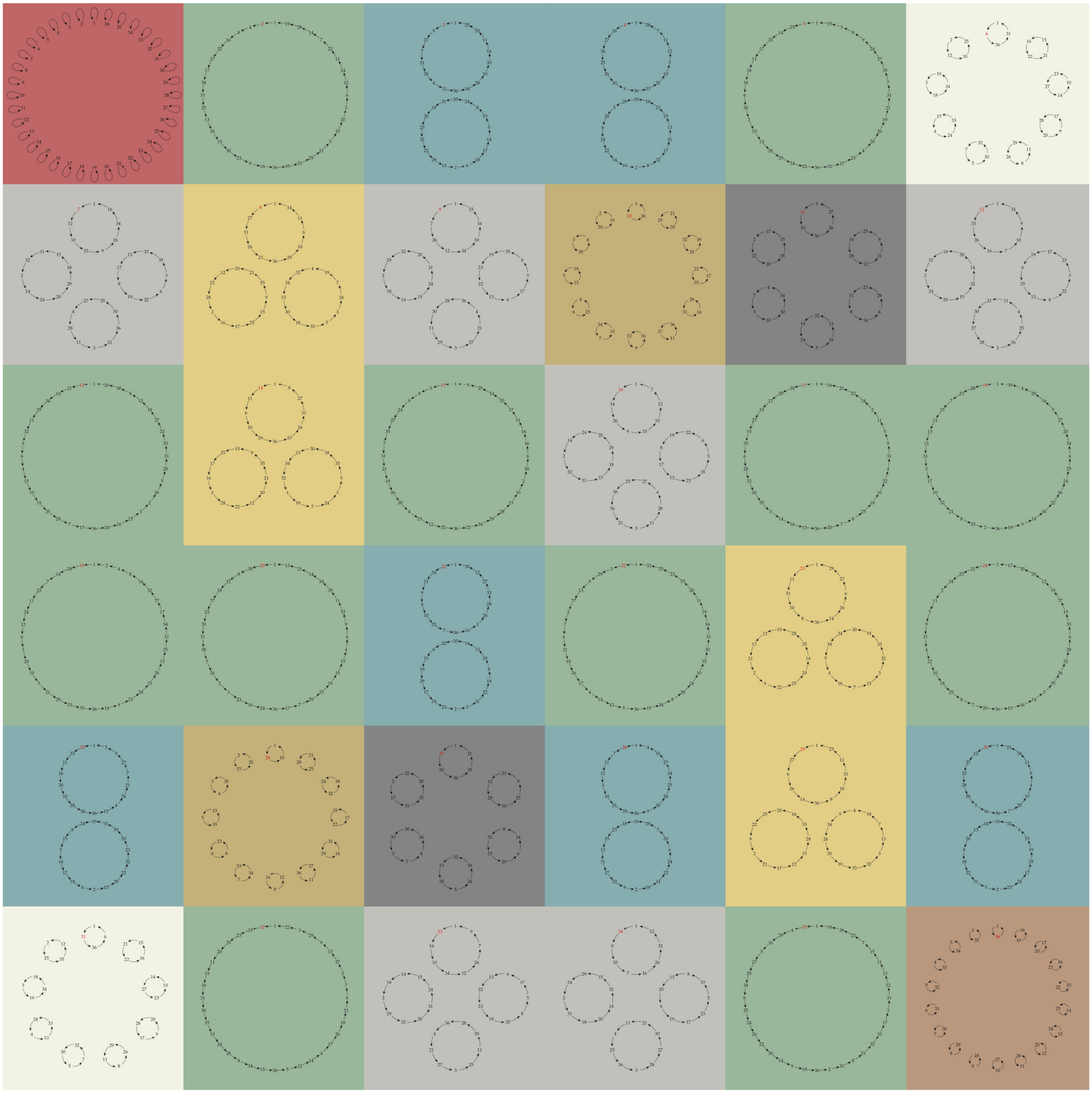


Figure 8. 36 Epicycles, by Martin H. Weissman

Remainders are the cold leftovers of arithmetic, but they are the tastiest morsels for mathematicians. In fact, remainders play a fundamental role in computer security. So what are these circles you see here? Zoom in on the big circle in the green tile, near the top-left. Starting at the top, the numbers are 1, 2, 4, 8, 16, 32. But the next number isn't 64. Since 64 ÷ 37 = 1, with a remainder of 27, the next number is the remainder: 27. Then you double again to 54. Since 54 ÷ 37 = 1, with a remainder of 17, the next number is 17. And so on— just keep on doubling, and every time you see a number bigger than 37, divide by 37 and take the remainder. The big circle shows you that this process eventually takes you back to 1.

On the cover, "Gaussian Fireworks," by Ellen Eischen.

Which shapes can you draw with an unmarked straightedge and compass, if you require all the edges to be the same length and the angles all to be equal to each other? This question stumped the ancient Greeks and others for 2000 years. Using *Gaussian periods*, the numbers visualized here, Carl Friedrich Gauss and Pierre Wantzel finally determined when such shapes could be constructed. Gaussian periods continue to play important roles in mathematics. Because the possibility of seeing their large-scale plots is so new, though, many mathematicians are not yet aware of their striking visual properties.

The Needell in the Haystack[1]

Getting to the Bottom of It: Gradient Descent and its Variants
by Anna Ma | edited by Jennifer Sidney Silva

Suppose you're standing at the top of a hill and your goal is to get to a box at the bottom of the hill. It's a nice, clear day and there are no obstacles in your way. You look out to the *landscape* of the hill and see the optimal, most direct path to get to the bottom. You take that path and open the box to find a map to a treasure chest. On the second day, you follow the map and find yourself at the top of another hill. According to the map, the treasure chest is at the bottom of this hill. Unfortunately, it's a very foggy day, so you cannot look out into the landscape to determine what direction to follow to get to the bottom of the hill, as you did before. This doesn't stop you, though; instead of using the global landscape, you decide to gather *local* information by looking at the ground around you. Using that to determine which direction the hill is sloping downwards, you take small steps, re-evaluating periodically the best direction to move in, until you eventually get to the bottom of the hill. There, you open the treasure chest to find one last map and a blindfold. The last map leads you again to the top of a hill, where you find a turkey that challenges you to get to the bottom of the hill blindfolded. However, in addition to being blindfolded, after every step you take you must spin around to face in a random direction. Now, how are you going to get to the bottom of the hill?

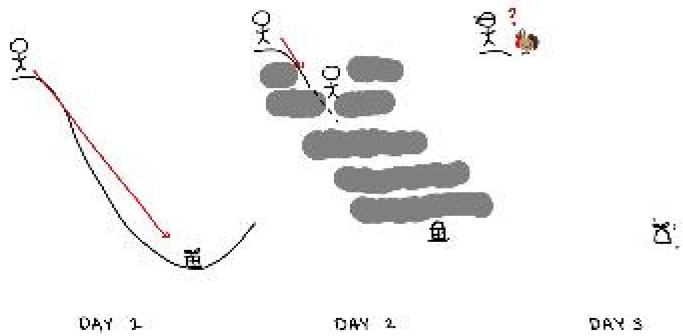


Figure 1. The landscape for each day.

In each of these examples, we start with the same problem (being at the top of a hill). But with each passing day, obtaining our goal becomes more difficult as information disappears. This is typically how we progress through difficult problems: you start with a simple problem, and then remove assumptions (such as access to the entire landscape) to see whether you can still solve it. In this article, we will journey through each problem as an analogy of different ways to minimize an objective function $F(x)$. We will then connect this to the previous installment of *The Needell in the Haystack* on the randomized Kaczmarz algorithm.

Minimizing the Objective Function on a Clear Day

Given a function $F(x)$, suppose we want to find a value x such that $F(x)$ is as small as possible. In Figure 1, we've placed our hill on an xy-plane. The graph of the function $F(x)$ looks like the hill and if we want to get to the bottom of the hill, we need to find x where $F(x)$ is as small as possible. Now, if we're on the first day, we know the entire landscape of the problem and we can compute directly the minimizer of $F(x)$. We can interpret this as knowing the

[1] This content supported in part by a grant from MathWorks. Anna Ma is a former student of Deanna Needell's.

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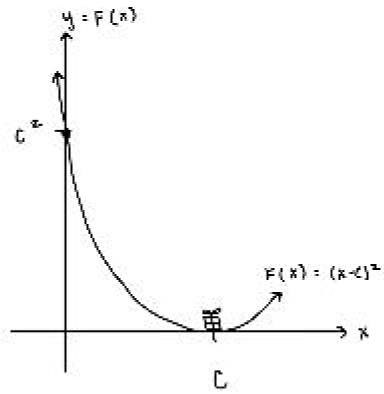

Figure 2. The hill as an objective function.

function $F(x)$ that describes the hill, and being able to compute a direct solution. In Figure 2, we have $F(x) = (x - c)^2$. From here we can deduce that the minimizer will be $x = c$ as $F(c) = 0$, whereas $F(x) \geq 0$ for all x, and we proceed directly to the bottom of the hill.

In higher dimensions, suppose we want to solve a linear system $Ax = y$ where x is an n-dimensional vector, A is an $m \times n$ matrix, and y is an m-dimensional vector. To find a solution to this linear system, we can seek to minimize the **least squares error** objective function $F(x) \equiv \|Ax - y\|^2$, where $\|v\|^2 = \sum_{i=1}^{m} v_i^2$, for v in R^m, denotes the squared Euclidean norm. Let $\hat{x} = \min_x F(x)$. This objective is referred to as the least squares error objective because we seek to minimize the **squared error**, or the sum of the squares of the differences between measurements y_i and inner products $\langle \hat{x}, a_i \rangle$ for an estimated $\hat{x}$, where a_i is the ith row of A. On a clear day, when the entire landscape information is accessible to us, we can solve for $\hat{x}$ directly. In particular, we can compute $\hat{x} = A^{\dagger} y$ where $A^{\dagger}$ denotes the Moore-Penrose pseudoinverse[2] and this will be the minimizer of $F(x)$. In fact, if we assume that our linear system has a unique solution, then $F(\hat{x}) = 0$. Now that we know how to solve our problem on nice, clear days, let's move on to foggy days.

Minimizing the Objective Function on a Foggy Day

On foggy days, we can only use local information to determine the direction we move in. Recall that on this day, our approach was to look at the ground around us to determine what direction was downhill, then take a few steps in that direction before looking around again to reevaluate the direction we should move in. This is exactly what the **gradient descent** algorithm does. More precisely, the gradient descent algorithm is an iterative process that updates successive approximation of minimizers of an objective $F(x)$ and proceeds as follows:

$$x_{t+1} = x_t - \alpha \nabla F(x_t),$$

where x_t denotes the t-th approximate minimizer of $F(x)$, α is a user-defined parameter that controls how drastically the approximations change from iteration to iteration, and $\nabla F(x_t)$ is the gradient of F at x_t, which is a vector that points in the direction that locally results in the biggest increase in the value of $F(x)$ and whose magnitude is a measure of how quickly $F(x)$ changes when you head in that direction. The parameter α is often referred to as the learning rate or step size. In our analogy, you are located at the approximate solution x_t, the learning rate α is a proxy for the length of the steps you're taking, and the local information you're using appears in the gradient descent update through the gradient evaluation $\nabla F(x_t)$.

For our example in Figure 2 with $F(x) = (x - c)^2$, the gradient of $F(x)$ is $2(x - c)$, thus our gradient descent algorithm prescribes taking the following iterates: $x_{t+1} = x_t - 2\alpha(x_t - c)$. When $\alpha = 1/2$, we can see that $x_{t+1} = c$. If we pick a smaller step size, it will take us more iterations to

[2] The Moore-Penrose pseudoinverse is a generalization of the matrix inverse when the matrix is non-square. When the matrix A is full rank, the pseudoinverse of A times itself is the identity matrix.

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get close to the minimizer $x = c$. If we take too large of a step size, we may overshoot and miss $x = c$, which can cause a delayed convergence. In some cases, it may even cause the algorithm to diverge. Figure 3 demonstrates both possibilities where α is chosen to be small (red arrows) and where α is chosen to be large (orange arrows).

This observation brings us to an interesting question: How do you choose the learning rate parameter α for gradient descent? The easiest way to pick α is to pick a small, conservative α and keep α the same, or fixed, for every iteration. We can also choose to start with a large value of α and slowly decrease α as the iterations progress. This would hopefully prevent us from experiencing the scenario presented in Figure 3. The learning rate can be selected *adaptively*, based on how steep the hill is at each iteration or other information (such as if you could use Google Maps to gain some additional information about your location). This is by no means an exhaustive list of ways to select the learning rate, and parameter selection is still an active area of research in the machine learning and optimization communities.

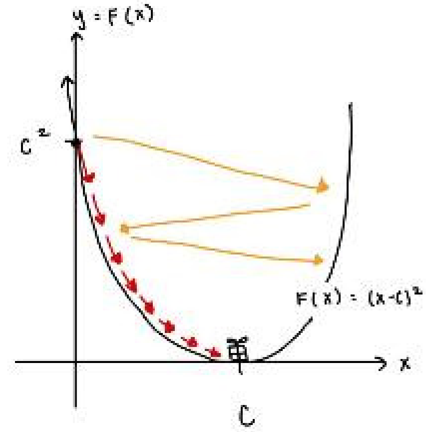

Figure 3. Different gradient descent paths based on a small learning rate (red) and a large learning rate (orange).

In the context of the least squares objective, the gradient can be written as $\nabla F(x) = A^T(Ax - y)$ and gradient descent prescribes the iterates $x_{t+1} = x_t - \alpha A^T(Ax_t - y)$. As in the 1-dimensional example, we can use the information from A and y to get a local approximation of the gradient but no longer have the resources to compute $A^\dagger$. What happens if we have even less information than that? In particular, what if we only have access to a single row a_i of A and its corresponding measurement y_i as in the randomized Kaczmarz algorithm? For this, we turn to our last setting.

Minimizing an Objective Function while Blindfolded

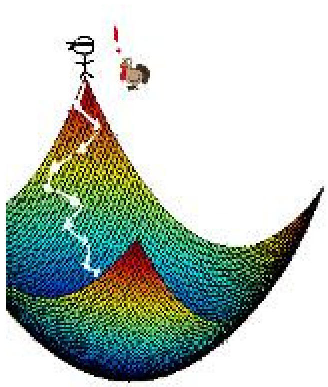

Figure 4. Example of a path for stochastic gradient descent.

When we are blindfolded, in addition to not being able to use local information around us (i.e., use the entire matrix A) to determine the direction the hill slopes downwards, we're also facing randomly chosen directions when making each step. Perhaps, however, we can feel an *approximate* direction in which the hill slopes downward with our feet. For any random direction we're facing (determined by a randomly selected row a_i), we will take a small step in the direction that feels like a downward direction and then continue with this iterative process. This is exactly how the **stochastic gradient descent (SGD)** algorithm works.

Instead of using the full gradient $\nabla F(x) = A^T(Ax - y)$, we consider approximations of the gradient of $F(x)$. When $F(x)$ can be written as a sum of some component functions, i.e., we can write $F(x) = \sum_{i=1}^{m} f_i(x)$, the iterations of SGD are written as $x_{t+1} = x_t - \alpha \nabla f_i(x_t)$, where $f_i(x)$ is a randomly chosen component of the objective function $F(x)$. If $\nabla f_i(x_t)$ are **unbiased** approximations of $\nabla F(x)$ (that is, the expected value of $\nabla f_i(x_t)$ is $\nabla F(x)$), then on average, we will be moving in the direction of the full gradient.

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For the least squares objective function, we can write $F(x) = \sum_{i=1}^{m} f_i(x)$, where $f_i(x) = (\langle a_i, x \rangle - y_i)^2$. Note that $f_i(x)$ only depends on a single row of the matrix A. To visualize this, we will need to move into a 3-dimensional figure, as illustrated in Figure 4. Note that in Figure 4, we do not take the most direct path to get to the minimum of the function; nevertheless, our path on average moves us in the direction of the minimum.

Connections to Randomized Kaczmarz

Just as with the randomized Kaczmarz algorithm, SGD is particularly useful when working with large-scale linear systems. In order to use gradient descent, we would need to compute the gradient of $F(x)$ and therefore need access to the entire matrix A. However, due to hardware constraints on memory, we may not be able to load A into our computer's working memory all at once. Alternatively, rows of A representing data points may be streaming in over time, thus we cannot use the entire matrix at once. The similarities between the randomized Kaczmarz algorithm and SGD are not superficial. In fact, they stem from the fact that we can think of the randomized Kaczmarz algorithm as a special instance of the SGD algorithm in which the objective function is the least squares objective and the learning rate is proportional to the row norm of the randomly selected row. That is, if we pick the component function $f_i(x)$ with probability $\|a_i\|^2 / \|A\|_F^2$ and set the learning rate $\alpha = 1/\|a_i\|^2$, then the SGD iterate becomes

$$x_{t+1} = x_t - \frac{1}{\|a_i\|^2}(\langle a_i, x_t \rangle - y_i)a_i^T;$$

this is exactly the same iterate as used in the randomized Kaczmarz algorithm! This connection allows us to leverage theoretical guarantees that are known about SGD to better understand the randomized Kaczmarz algorithm, and vice versa. As with the Kaczmarz algorithm, many variants of the SGD algorithm have been proposed. If you're curious, you might look up one of the following: the adaptive gradient algorithm (Adagrad), root mean square error propagation (RMSprop), adaptive momentum estimation (Adam), or see [2] for an overview. However, these variations of SGD, unlike the Kaczmarz algorithm, are not used just for solving linear systems.

Beyond Least Squares Objectives

In some sense, the objective functions that we've been looking at so far have been fairly well-behaved. This "well-behavedness" is characterized by a concept we call **convexity**. A function is considered to be a convex function if for any x_1 and x_2, and for any $0 \le t \le 1$, we have

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$

Figure 5 illustrates this definition.

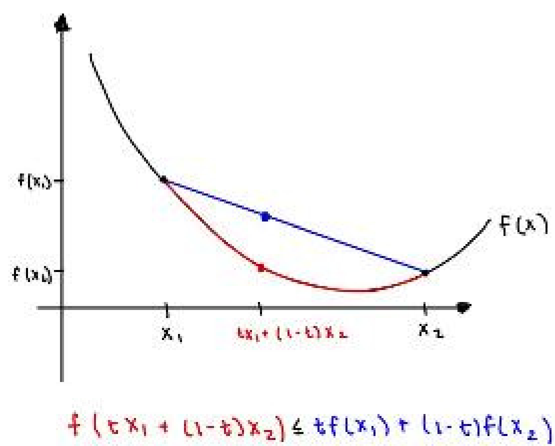

Figure 5. Convexity.

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Geometrically, this says that if we draw a line segment between any two points on a graph (blue), the graph of the function between (red) lies below the segment. Note that the functions in Figures 2-4 are examples of convex functions. The least squares objective is also an example of a convex function. The reason convex functions are considered well-behaved is because their minimum values are global minimums. The beauty of SGD and gradient descent algorithms is that you can use them to solve many different kinds of objective functions; they're not restricted to the least squares objective as the Kaczmarz algorithm is, nor are they restricted to convex functions!

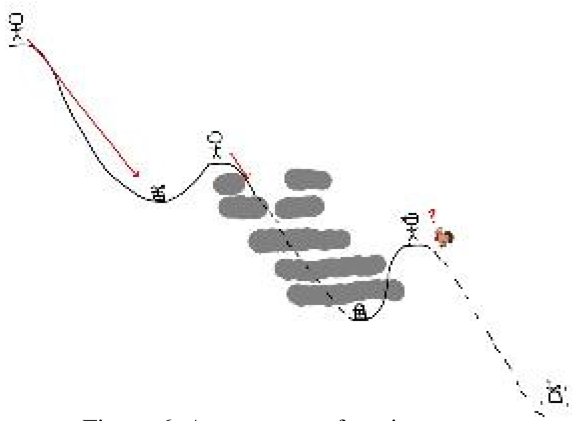

Figure 6. A nonconvex function.

A function that is not convex is referred to as a **nonconvex** function. If our function is convex, it usually has a single, unique minimizer, but if we have a nonconvex function, there could be small valleys that we get stuck in when using gradient-based methods. Take, for example, the function illustrated in Figure 6. In this figure, we have zoomed out of our single hill and represented the objective as a function of all three hills we descended upon during our journey in the introduction. The first two valleys represent **local minima**, while the last valley represents our global minimum. We can also see that the hilly landscape in Figure 6 does not satisfy the definition of convexity, thus is a nonconvex objective function. Nonconvexity can make minimizing problems very difficult, as the landscape of the problem can become quite complex. Figure 7 shows one such real-world objective function that emerges when optimizing a special type of function called a convolutional neural network [3].

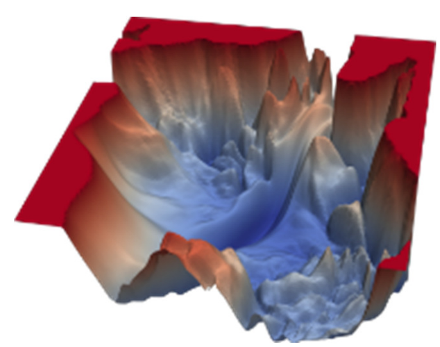

Figure 7. Nonconvex objective function for training a convolutional neural network.

Nonconvex optimization brings to light many interesting problems and questions. In both Figure 6 and Figure 7, the minimum we obtain using gradient-based methods depends heavily on our starting point. Hence, how we pick an initial point is an important question. And what would we do if we end up at a local minimum instead of a global minimum? How do we know if we are at the global minimum or a local minimum if we can't see the entire landscape? Are local minimums even meaningful? These questions and more are just waiting to be answered!

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[3] H. Li, Z. Xu, G. Taylor, C. Studer, and T. Goldstein, Visualizing the loss landscape of neural nets, *Advances in Neural Information Processing Systems* **31** (2018).

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Valentine's Math, Part 3

by Ken Fan | edited by Jennifer Sidney Silva

Emily and Jasmine concocted the equation

$$\sqrt{x^2+y^2} = 1 + \frac{1}{2}\frac{y|x|}{x^2+y^2}\left(1+\frac{1}{(7/4+y)^6}\right)$$

for their Valentine heart design. They remembered that their whole Valentine heart adventure began with a Valentine heart app[1] created by Jürgen Richter-Gebert. They decide to see what equation the app used for its Valentine heart.

Emily: The app designer, Mr. Richter-Gebert, conveniently provides the equation for his Valentine heart. It's

$$(x^2 + ((1+b)y)^2 + z^2 - 1)^3 - x^2z^3 - ay^2z^3 = 0,$$

where a and b are parameters. But we're only interested in the central cross section.

Jasmine: Actually, what directions do x, y, and z correspond to? We need to find out so that we can obtain the correct 2D cross section.

Emily: He doesn't say.

Jasmine: I guess we'll have to figure it out.

Emily: The Valentine heart has symmetries that should be reflected in the equation. It has two planes of mirror symmetry.

Jasmine: Good idea! In fact, the variables x and y only appear raised to even powers in the equation. That means that if (x, y, z) is a solution, then so is $(\pm x, \pm y, z)$ for any choice of plus and minus signs. In other words, the planes $x = 0$ and $y = 0$ are planes of mirror symmetry.

Emily: And the last two terms in the equation have z raised to an odd power.

Jasmine: So the z direction in the app corresponds to our y direction.

Emily: We still have to figure out whether to take the cross section of the Valentine heart by the plane $x = 0$ or the plane $y = 0$. One of these will split the heart into two lobes, while the other will yield the cross section we're looking for.

[1] Emily is looking at the widget at https://love.imaginary.org/ created by Jürgen Richter-Gebert. (Note: In Part 1 in Volume 14, Number 3, we misattributed the app to Aaron Montag. Mr. Montag designed the ray tracer used by Mr. Richter-Gebert to create the Valentine heart app. We apologize for any confusion this may have caused.)

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Jasmine: Let's set x and y to 0 in turn to get two equations, and decide which one produces a heart shape. If we set x to 0, we get

$$((1 + b)y)^2 + z^2 - 1)^3 - ay^2z^3 = 0,$$

and if we set y to 0, we get

$$(x^2 + z^2 - 1)^3 - x^2z^3 = 0.$$

Emily: That's strange! When you set y to 0, you end up with an equation that does not depend on the parameters a and b at all. That means we can take a shortcut to figure out which cross section corresponds to the Valentine heart shape and which corresponds to the lobe separator. All we have to do is fiddle with the app parameters and see which cross section doesn't change. The one that doesn't change will be the one that corresponds to setting y to 0!

Emily and Jasmine move the sliders on the app.

Jasmine: It's pretty clear that when you change the parameters, it's the lobe-separating cross section that morphs. When b is really small, the top of the Valentine heart starts to look more like a Fuji apple!

Emily: In fact, if we set $b = 0$, the lobe-separating cross section equation becomes

$$(y^2 + z^2 - 1)^3 - ay^2z^3 = 0,$$

which only differs from the $y = 0$ cross section in that it has the multiplicative parameter a in the last term and replaces x with y. So when $a = 1$, the two cross sections *will* be congruent.

Jasmine: If we set a to 1 and b to 0 in the original equation, we get

$$(x^2 + y^2 + z^2 - 1)^3 - x^2z^3 - y^2z^3 = 0.$$

Look! That's rotationally symmetric! Because $x^2 + y^2$ is the square of the distance of a point from the z-axis, and if we make the substitution $r^2 = x^2 + y^2$, the equation becomes

$$(r^2 + z^2 - 1)^3 - r^2z^3 = 0.$$

This shows that if (x, y, z) is a solution, then so are all the points on the horizontal circle centered at $(0, 0, z)$, which passes through that point.

Emily: So when $a = 1$ and $b = 0$, we get the surface of revolution obtained by rotating the Valentine heart about the z-axis! I guess that means the shape, unlike a Fuji apple, should still have a little peak at the bottom.

Emily and Jasmine rotate the object in the app to reveal its bottom; sure enough, they see a little point sticking out.

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Jasmine: Still, we lucked out! The cross section we're interested in is the one that corresponds to setting y to 0, and since that equation does not involve the parameters a and b, we don't even have to be concerned with them.

Emily: Let's rewrite the cross section corresponding to $y = 0$, but let's replace z with y to bring the notation into alignment with the notation we originally used to produce our Valentine heart.

Jasmine: Okay, then the equation is

$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0.$$

Emily: It's quite a bit simpler than the equation we came up with! It doesn't have radicals or absolute values.

Jasmine: In fact, it's the zero set of a polynomial. Why does this produce a Valentine heart?

Emily: Hmm. We can rewrite the equation as

$$(r^2 - 1)^3 - x^2y^3 = 0,$$

where $r^2 = x^2 + y^2$; now that we've changed z to y, this simply becomes the square of the distance to the origin. So $(r^2 - 1)^3$ is a kind of measure of the deviation from the unit circle centered at the origin. When $(r^2 - 1)^3$ is 0, the point is on the unit circle. A point (x, y) will be inside or outside this unit circle depending on whether $(r^2 - 1)^3$ is negative or positive, respectively.

Jasmine: This is not so different from what we were doing. We also started with a unit circle and modified it to make it look more like a Valentine heart, although we were more direct. Our equation can be written

$$r - 1 = \frac{1}{2}\frac{y|x|}{x^2 + y^2}\left(1 + \frac{1}{(7/4 + y)^6}\right).$$

If we had simply $r - 1 = 0$, then we would have an equation whose graph *is* the unit circle centered at the origin, so all the stuff on the right-hand side of this equation expresses how the Valentine heart shape deviates from the unit circle.

Emily: By contrast, the app's equation can be written $(r^2 - 1)^3 = x^2y^3$. So instead of specifying how the more direct "$r - 1$" is modified, the app is specifying the deviation from the unit circle with the more complicated function of the "deviation distance" $(r^2 - 1)^3$.

Jasmine: The deviation x^2y^3 has x raised to the even power 2, which means that the deviation is symmetric about the y-axis, thus producing a Valentine heart with bilateral symmetry. That's good!

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Emily: If we fix x and let y vary, then x^2y^3 strictly increases as y goes from negative infinity to positive infinity, going through 0 when y is 0.

Jasmine: That's also similar to our equation! That means the app's Valentine heart will bulge from the unit circle in the upper half-plane and pull in from the unit circle in the lower half-plane. And because x^2y^3 is 0 on the axes, the app's Valentine heart, just like ours, will intersect the unit circle on the axes.

Emily: I think it might be fruitful to convert everything to polar coordinates.

Jasmine: Okay, let's do that!

Emily: Since the Valentine heart is symmetric about the y-axis, let's use r for the distance from the origin; but let's let θ be the angle as measured clockwise from the positive y-axis, instead of using the usual convention that θ is the angle measured counterclockwise from the positive x-axis.

Jasmine: Okay. In that case, the relationship between (x, y) and (r, θ) would be $x = r \sin \theta$ and $y = r \cos \theta$ (instead of the conventional $x = r \cos \theta$ and $y = r \sin \theta$). Substituting these into the equation, we get

$$(r^2 - 1)^3 = (r \sin \theta)^2(r \cos \theta)^3 = r^5 \sin^2 \theta \cos^3 \theta.$$

Emily: By symmetry, we can restrict our attention to values of θ between 0° and 180°.

Jasmine: For those values, $\sin^2 \theta$ starts off flat at $\theta = 0°$, rises to 1 at $\theta = 90°$ – where it is again flat – then symmetrically returns to 0 at $\theta = 180°$. The profile is like that of a gentle hill.

Emily: Meanwhile, $\cos^3 \theta$ starts off flat at $\theta = 0°$, then decreases until it flattens out as it passes through 0 at $\theta = 90°$, then symmetrically decelerates until it reaches -1 at $\theta = 180°$, where it is flat again.

Jasmine: So when we multiply to get $\sin^2 \theta \cos^3 \theta$, we will get two identical, gentle hill profiles: one will be between 0° and 90°, and the other will be between 90° and 180° – though the one between 90° and 180° is a flipped version of the hill between 0° and 90°.

Emily: The values of $\sin^2 \theta \cos^3 \theta$ do not directly measure to the deviations from the unit circle. This is because on each radial line, the location where the Valentine heart intersects the radial line occurs at the distance r from the origin that satisfies $(r^2 - 1)^3/r^5 = \sin^2 \theta \cos^3 \theta$. At least we know that the bigger $\sin^2 \theta \cos^3 \theta$ is, the farther out from the unit circle this intersection will be; and the more negative $\sin^2 \theta \cos^3 \theta$ is, the farther in from the unit circle the intersection will be. So it makes sense that the result will have the general Valentine heart-shape look.

Emily and Jasmine modify their computer program to plot the shape.

Jasmine: It's nice, but I think I prefer more taper at the bottom.

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Emily: We know how to fix that!

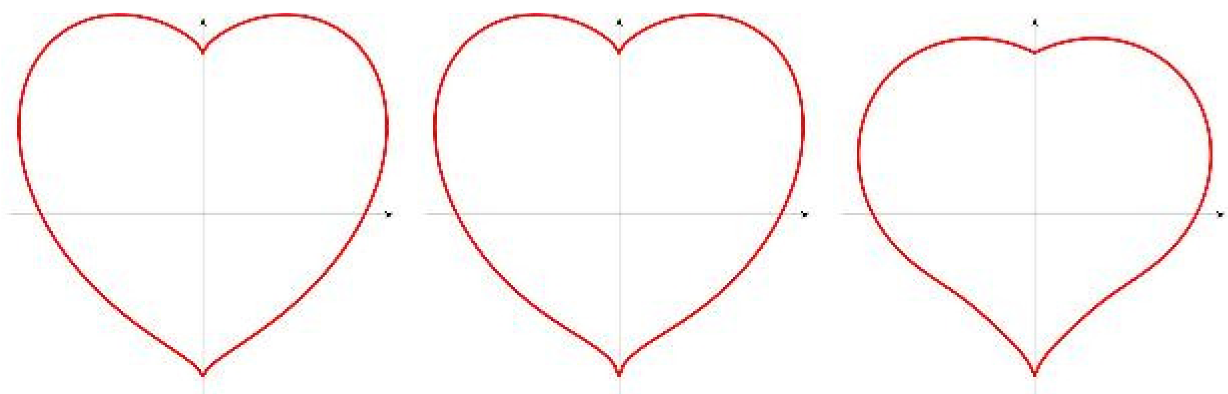
Jasmine: You’re right!

Emily: We can multiply that deviation by the factor we devised to increase the concavity of the taper in our design.

Jasmine: Let’s do that! So the equation would become

$$(x^2 + y^2 - 1)^3 - x^2 y^3 \left(1 + \frac{1}{(7/4 + y)^6}\right) = 0 .$$

Emily and Jasmine plot this version of a Valentine heart flanked by the one in the app and the one that they created.



$(x^2 + z^2 - 1)^3 - x^2 z^3 = 0$ $\quad$ $(x^2 + y^2 - 1)^3 - x^2 y^3 (1 + \frac{1}{(7/4 + y)^6}) = 0$ $\quad$ $\sqrt{x^2 + y^2} = 1 + \frac{1}{2} \frac{y|x|}{x^2 + y^2} (1 + \frac{1}{(7/4 + y)^6})$

Emily: I’m not sure which one I like best.

Jasmine: Me neither. I guess it depends on my mood.

Emily: Anyway, everybody’s heart is unique!

If you come up with an equation for a Valentine heart, we’d love to know. Please email it to us at girlsangle@gmail.com.

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Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 29 - Meet 1 September 9, 2020	Mentors:	Mandy Cheung, Cecilia Esterman, Jenny Kaufmann, Bridget Li, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Rachel Zheng

It's been inspirational to us that our members continue to want to do math despite our having to continue meeting virtually. Doing math is the best way to learn math, and to do it, find, or, better, ask math questions that intrigue you and for which you feel you can act upon. If you have a math question that doesn't intrigue you or you cannot act upon, try to adjust it and make it something that you enjoy thinking about.

Also, a huge Thank You to all our mentors for pressing on in this very challenging virtual environment.

Session 29 - Meet 2 September 16, 2020	Mentors:	Mandy Cheung, Bridget Li, Yuyuan Luo, Vievie Romanelli, Sakshi Suman, Jane Wang, Angelina Zhang, Rachel Zheng

There are so many variants on the task of creating mathematical expressions involving given numbers that evaluate to some target value. One variant which arose led to the following question: Given positive integers N and T, what is the least number of copies of N needed in an expression that involves addition, subtraction, multiplication, division, parentheses, and only the number N that evaluates to T? Even when $N = 1$, the answer is not so straightforward and is related to another variant on this sort of question which **Allie** explored and wrote up in her article *The Greatest Number From N Ones* on pages 22-25 of Volume 11, Number 4 of the Girls' Angle Bulletin.

Session 29 - Meet 3 September 23, 2020	Mentors:	Mandy Cheung, Jenny Kaufmann, Yuyuan Luo, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Angelina Zhang

Exploring geometric sequences modulo N is a wonderful way to begin exploring modular arithmetic. Such sequences are necessarily periodic. What are the periods? What is the longest period, and when is this longest period achieved?

Session 29 - Meet 4 September 30, 2020	Mentors:	Mandy Cheung, Cecilia Esterman, Jenny Kaufmann, Bridget Li, Tina Lu, Yuyuan Luo, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Angelina Zhang

Related to the expression forming of Meet 2, some members set the goal of writing the smallest positive number they could muster using N fives, addition, subtraction, multiplication, division, exponentiation, and parentheses, for various values of N. For $N = 5$, can you beat

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$$5^{-5^{5^5}}$$

(Note that when written without parentheses, $a^{b^c} = a^{\left(b^c\right)}$.)

Session 29 - Meet 5 October 7, 2020	Mentors:	Mandy Cheung, Cecilia Esterman, Jenny Kaufmann, Bridget Li, Tina Lu, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Angelina Zhang

Mentor Cecilia is overseeing a group exploring patterns in Pascal's triangle. Rather than showing them the patterns, she's giving them free reign to find patterns that intrigue them, and then they work on understanding and proving them. In fact, Cecilia didn't even introduce Pascal's triangle, because the whole saga began with the question of expanding $(a + b)^2$, and following that up with the natural next problems of expanding $(a + b)^3$, $(a + b)^4$, etc. In this way, the members she's working with are gaining experience in the process of doing math and developing their own unique perspective on Pascal's triangle. Because the members are approaching it in their own unique way, they've been free to consider how it might make sense to extend the triangle to "negative rows."

Session 29 - Meet 6 October 14, 2020	Mentors:	Cecilia Esterman, Jenny Kaufmann, Bridget Li, Tina Lu, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Sakshi Suman, Jane Wang, Rebecca Whitman

Working with mentor Kate, some members are creating geometric designs using Python's turtle graphics library.

Session 29 - Meet 7 October 21, 2020	Mentors:	Cecilia Esterman, Jenny Kaufmann, Bridget Li, Kate Pearce, AnaMaria Perez, Vievie Romanelli, Sakshi Suman, Jane Wang, Rebecca Whitman, Rachel Zheng

Calculus enables us to readily compute the slopes of the tangent lines to the graph of the parabola $y = x^2$. In fact, many students simply memorize that the derivative of x^n with respect to x is nx^{n-1}. However, how many ways can you think of to find the slopes of the lines tangent to the parabola $y = x^2$ *without using calculus*?

Session 29 - Meet 8 October 28, 2020	Mentors:	Cecilia Esterman, Bridget Li, Tina Lu, Kate Pearce, AnaMaria Perez, Laura Pierson, Vievie Romanelli, Sakshi Suman, Rebecca Whitman, Angelina Zhang, Rachel Zheng

One of our members explored the standard rules of differential calculus to understand the rules better and gain intuition for them. For example, consider the quotient $(ax + b)/(cx + d)$, where a, b, c, and d are constants. Applying the formula for an infinite geometric series, this can be rewritten as $b/d + (ad - bc)/d^2 \cdot x$ + higher order terms. We claim that looking at the coefficient of x gives us all we need to know to be able to write down the quotient rule. Do you see?

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Calendar

Session 29: (all dates in 2021)

September	9	Start of the twenty-ninth session!
	16	
	23	
	30	
October	7	
	14	
	21	
	28	
November	4	
	11	Karia Dibert, University of Chicago
	18	
	25	Thanksgiving - No meet
December	2	

Session 30: (all dates in 2022)

January	27	Start of the thirtieth session!
February	3	
	10	
	17	No meet
	24	
March	3	
	10	
	17	
	24	No meet
	31	
April	7	
	14	
	21	No meet
	28	
May	5	

Girls' Angle has been hosting Math Collaborations at schools and libraries. Math Collaborations are fun math events that can be adapted to a variety of group sizes and skill levels. We will soon have versions available that are designed for remote participation. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

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Girls' Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ______________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don't like math, what don't you like? If you love math, what do you love? What would you like to get out of a Girls' Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls' Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

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Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

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Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, founder and director of the Exploratory
Yaim Cooper, Institute for Advanced Study
Julia Elisenda Grigsby, professor of mathematics, Boston College
Kay Kirkpatrick, associate professor of mathematics, University of Illinois at Urbana-Champaign
Grace Lyo, assistant dean and director teaching & learning, Stanford University
Lauren McGough, postdoctoral fellow, University of Chicago
Mia Minnes, SEW assistant professor of mathematics, UC San Diego
Beth O'Sullivan, co-founder of Science Club for Girls.
Elissa Ozanne, associate professor, University of Utah School of Medicine
Kathy Paur, Kiva Systems
Bjorn Poonen, professor of mathematics, MIT
Liz Simon, graduate student, MIT
Gigliola Staffilani, professor of mathematics, MIT
Bianca Viray, associate professor, University of Washington
Karen Willcox, Director, Oden Institute for Computational Engineering and Sciences, UT Austin
Lauren Williams, professor of mathematics, Harvard University

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

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Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is $216 for one session (12 meets) ☐ I will pay on a per meet basis at $20/meet.	☐ I will pay on a per meet basis at $30/meet. ☐ I'm including $50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

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Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ___________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____

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