Girls' Angle Bulletin

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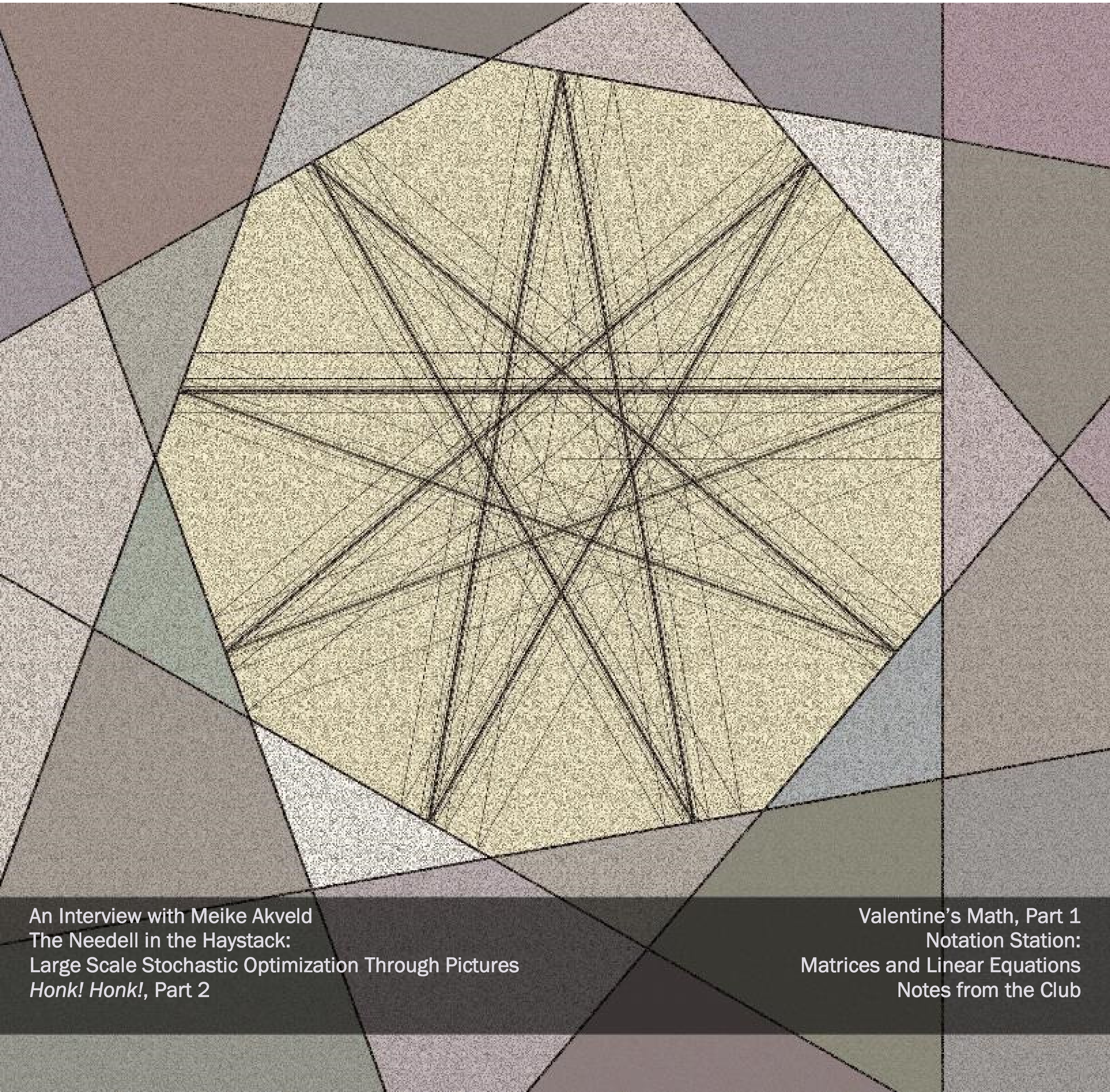
To Foster and Nurture Girls' Interest in Mathematics



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From the Founder

There's math in all our members – in everybody. Time and time again, my faith in that has proven true. I have ideas for how to draw it out, but I have no idea when the math will appear or what form it will take. When it happens, it's always thrilling. - Ken Fan, President and Founder

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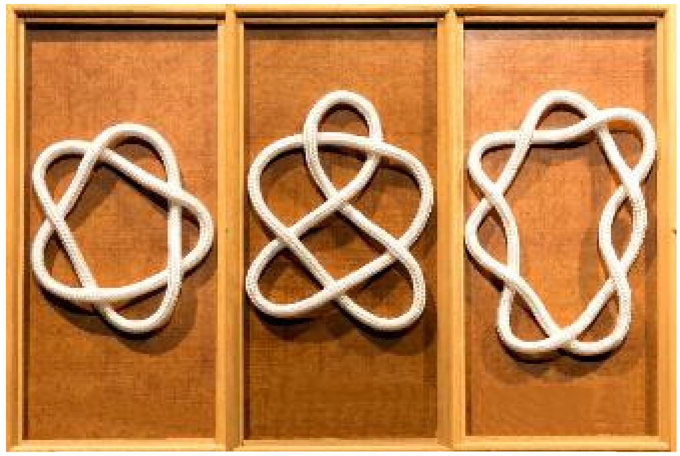
Girls' Angle: A Math Club for Girls

The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.

On the cover: "Kaczmarz Star" by C. Kenneth Fan. For the math underlying this image, see Needell in the Haystack, page 9.

An Interview with Meike Akveld



Some knots. Image modified from a Wikimedia Commons image attributed to Matemateca (IME/USP)/Rodrigo Tetsuo Argenton.

Meike Akveld is a Senior Scientist at ETH Zürich. She received her Doctor of Philosophy in Mathematics from ETH Zurich under the supervision of Leonid V. Polterovich and Dietmar Arno Salamon.

On February 9, 2012, Girls' Angle was fortunate to enjoy a visit from her as part of our Support Network. She discussed knots. For more details of her visit, see pages 29-30 of Volume 5, Number 3 of this Bulletin.

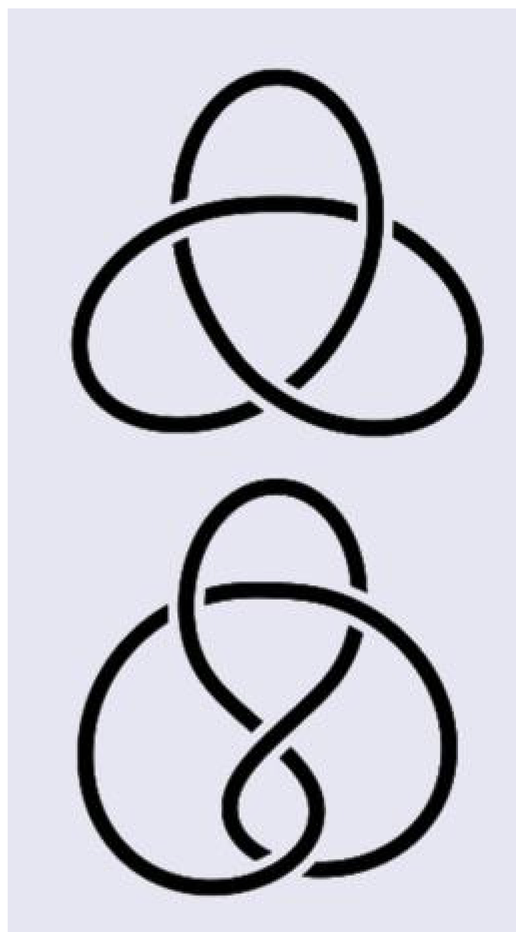
Ken: What is your favorite kind of mathematics?

Meike: Ah, that's an interesting question. My background is in symplectic geometry, so I love (differential) geometry and topology. In particular knots are one of my favourite objects. But I also love certain parts of algebra and analysis. I think what makes mathematics interesting to me is the beauty. I don't like tedious computations, but if there is beauty to be had, then I am on board.

Ken: What are knots and what do you find intriguing about them?

Meike: Let me first answer the first part of the question. What are knots? Well, everyone knows what knots are. You take a piece of rope and you make a knot in it and there you go, a knot. (See the image of three knots in the upper right corner.)

They are useful for climbing or in sailing, but also for simple daily problems. And they can be beautiful from an aesthetic point of view. That's the knots mathematicians are looking at. Here are two simple knots drawn in an abstract way (in order to deal with knots, mathematicians look at so-called knot diagrams, which are a projection of the knot on a 2-dimensional plane):



Dear Reader,

We're committed to producing quality math educational content and make every effort to provide this content to you for free. We are also committed to surviving as a nonprofit!

For this issue, those who do not subscribe to the print version will be missing out on a portion of this interview with Prof. Meike Akveld and other content. We hope that you consider the value of such content and decide that the efforts required to produce such content are worthy of your financial support.

We know that mathematical interest and talent is unrelated to economic status, which is why we provide so much content for free. But we hope that those of you who are comfortable financially will help us to continue in our efforts.

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Thank you and best wishes,
Ken Fan
President and Founder
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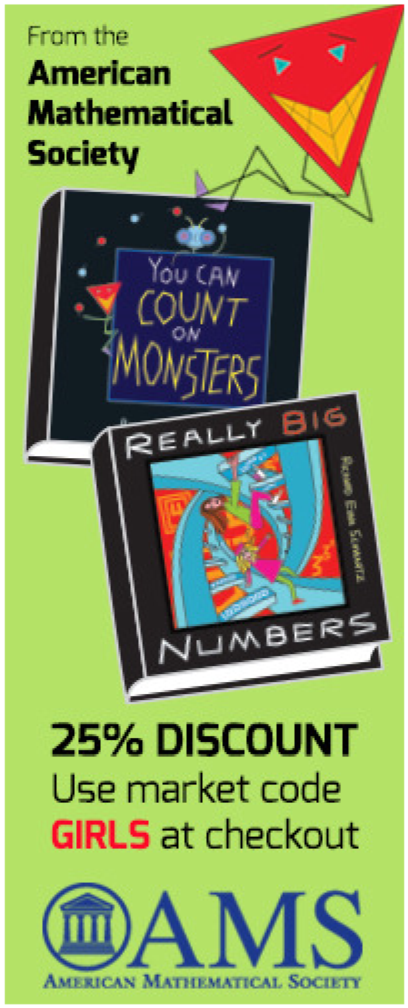

The American Mathematical Society is generously offering a 25% discount on the two book set *Really Big Numbers* and *You Can Count On Monsters* to readers of this Bulletin. To redeem, go to http://www.ams.org/bookstore-getitem/item=MBK-84-90 and use the code "GIRLS" at checkout.

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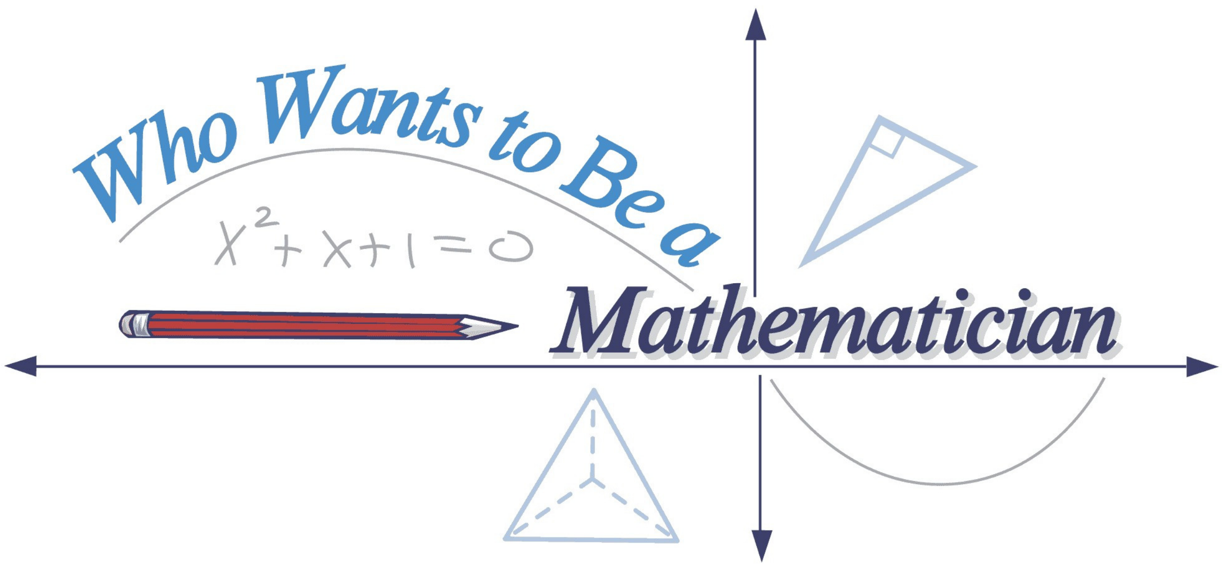

America's Greatest Math Game: Who Wants to Be a Mathematician.

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The Needell in the Haystack[1]

Large-Scale Stochastic Optimization through Pictures
by Deanna Needell | edited by Jennifer Sidney Silva

Big data brings many new challenges. In a recent installment,[2] I wrote a bit about bias and gave some examples like Simpson's paradox, which can be seen even with small amounts of data. When the data is large, bias becomes still harder to detect; indeed, machine learning methods are used most often because the amount of data is so large that it is impossible for humans to go through it all and make conclusions. But if humans cannot go through the data, how can we determine if the output is reasonable or biased? This is but one example of the challenges brought forth by the large-scale nature of modern data.

In this article, I will discuss methods that address another challenge, namely, the problem that occurs when the data is so large it can't even be fully loaded into computer memory. How does one analyze the data in this case?

Parallelization

Our first thought may be to distribute the data over many machines, and to divide the workload accordingly. Indeed, this is a common practice which is known as **distributed** or **parallel computing**; the process of distributing a task over several machines is known as **parallelization**. However, it is not always easy to "divide and conquer," as some methods lend themselves easily to parallelization while others do not. In fact, some problems are coined **embarrassingly parallel**, and – as the name suggests – are quite straightforward to break into parallel computing tasks while maintaining efficiency. The author, while smiling at the coinage, disagrees that there is anything "embarrassing" about being able to parallelize oneself; she prefers the terms *perfectly parallel, delightfully parallel,* or even the alliteration *pleasingly parallel*. If you have recently played a video game that uses a graphics processing unit, you have likely experienced an implementation of a *pleasingly parallel* algorithm for 3D image rendering. There isn't a definitive line separating methods that are easily parallelizable and those that are not, but it is believed that there are indeed some that are truly "harder" to make parallel efficiently; an example is the "N-body problem," which asks for the motion of N astronomical bodies as they travel under the influence of each other's gravitational pull.

If you like Snow White, or are interested in learning more about parallelization, I suggest perusing a 2006 report entitled "The Landscape of Parallel Computing Research: A View from Berkeley," by Asanovic et al. The report details the "13 dwarfs" of parallel computing, which are 13 problems believed to be widely applicable and distinct enough to represent classes of problems that may or may not lend themselves well to parallelization.

Really BIG Matrices

In this article, we will consider an example of a situation where the data is so big that only pieces of it can be held in the computer's memory at any given moment. The example has interesting research directions to explore and enjoys a wide array of modern applications. The method also lends itself well to visualization. The problem consists of solving a linear system of

[1] This content supported in part by a grant from MathWorks.
[2] See Volume 13, Number 5.

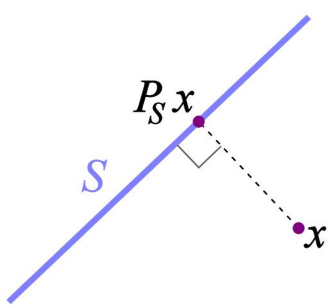
equations $Ax = b$, but as I mentioned before, we consider the setting in which the matrix A is so large that it doesn't fit into computer memory. Specifically, let us suppose that the m by n matrix A is very *overdetermined*, so that the number of rows m is far larger than the number of columns n, and that m is so large that doing operations on the columns of A – let alone the entire matrix itself – is impossible on your machine. In these circumstances, how do you solve such a system?[3]

If you have taken a linear algebra course, then you have learned that there are several ways to solve such systems, including things like Gaussian Elimination, computing certain factorizations, or even computing the (pseudo-)inverse of A and then applying it to b. So which of these would be easiest to do if, for example, you can only access a single row of A at a time?

Kaczmarz Method

A Polish mathematician named Stefan Kaczmarz[4] came up with a method in 1937 that would turn out to provide a nice approach to this problem (though in 1937, he likely had no idea of its role in parallelization or large-scale computing). Now called the Kaczmarz method, this method saw a spark of revival a few decades ago when large-scale computing sought efficient methods that were easy to implement without needing to access the full set of data at once.

Here's how the method goes: given an initial starting guess x_0, select a row a_i of A and get a new approximation x_1 by projecting x_0 onto the solution space $\langle a_1, x\rangle = b_i$ given by that row. Repeat this process using another row of A, thereby creating a sequence of approximations $x_0, x_1, x_2, \ldots$. The original method developed by Kaczmarz used the rows of A in a cyclic fashion, namely, first projecting onto the space defined by the first row of A and then the second, and so on. The *random* Kaczmarz method instead randomly selects a row of A in each iteration, and there are many ways to make such row selections.



By construction, the method produces approximations x_i, each of which satisfies at least one of the linear equations in our system. But do these iterates converge to a solution of the entire system, and how are they computed? Let's address the latter question first. Recall that the orthogonal projection of a point x onto a space S is the point $P_S\, x$ in S that is closest (in the Euclidean norm) to the point x. The name *orthogonal projection* is appropriate since the vector pointing from x to $P_S\, x$ is orthogonal to the space S (see the picture to the left). The projection can be computed algebraically as well, leading to the following update rule (the initial guess, x_0, is chosen arbitrarily, and i_k is the index of the row chosen on the kth iteration):

$$x_{k+1} = x_k + \frac{b_{i_{k+1}} - \left\langle a_{i_{k+1}}, x_k \right\rangle}{\left| a_{i_{k+1}} \right|} a_{i_{k+1}} \ .$$

We can then verify that the iterate x_k satisfies the i_kth equation for $k > 0$, i.e.,

$$\left\langle a_{i_k}, x_k \right\rangle = b_{i_k} \ .$$

[3] In terms of Algebra I, we have a system of m linear equations in n unknowns, where m is much larger than n.

[4] The author does not speak Polish, but believes the pronunciation of Kaczmarz is similar to "Catch-marsh."

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Honk! Honk!, Part 2

An Introduction to Parking Functions[1]

by Kimberly P. Hadaway and Pamela E. Harris[2][3]

We are now ready[4] to unravel Pollak's proof (as presented in [7]) that $|PF_n| = (n + 1)^{n-1}$. Our method is to explain each sentence in more detail, following this with corresponding explanations as we work through the $n = 3$ case, which we discussed in Example 1.

Sentence 1: "Add an additional space n + 1 and arrange the spaces in a circle."

This sentence instructs us to lengthen our one-way street by adding an additional parking spot numbered $n + 1$. This sentence also instructs us to reshape our one-way street. In the diagram below, we can see that our original street is represented as a line, like most streets.

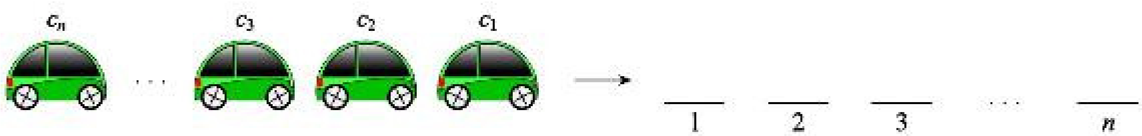

Figure 3. Straight street model.

At this point in the proof, we have restructured our street to be circular (as in the diagram below), and it is still a one-way street.

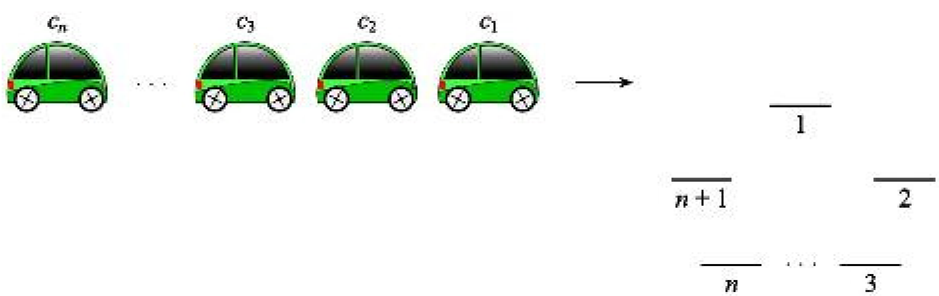

Figure 4. Circular street model.

Connection to Example 1: Instead of having 3 parking spots on a one-way street, we now have 4 parking spots arranged in a one-way circle. Here is what our straight street looks like at the beginning of this example.

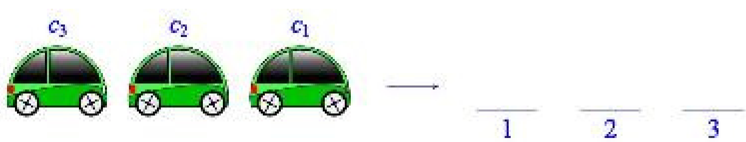

Figure 5. Straight street with $n = 3$.

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[1] This content is supported in part by a grant from MathWorks.

[2] Both authors are from the Department of Mathematics and Statistics at Williams College.

[3] The authors thank Miguel Lopez and Adam Martinson for pointing out an indexing typo in the proof of Lemma 3.

[4] For notation and definitions, please see *Honk! Honk!*, Part 1 in the previous issue of this Bulletin.

After following the instructions in Sentence 1, we have a circular street that looks like the one below.

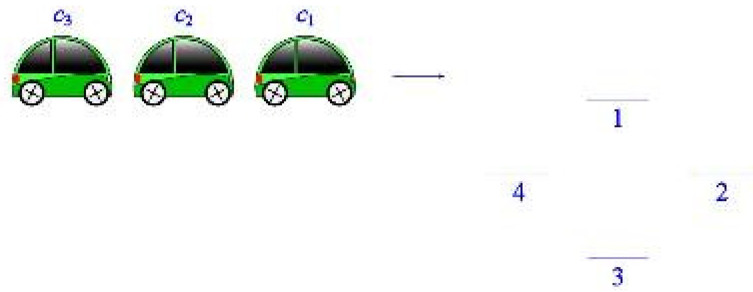


Figure 6. Circular street with $n = 3$.

Sentence 2: "Allow $n + 1$ also as a preferred space."

Here, we are just saying that we can allow cars to declare parking spot $n + 1$ as their preferred parking spot.

Connection to Example 1: Each car can now say that they want to park in spots 1 through 4.

Sentence 3: "Now all cars can park, and there will be one empty space."

Since we have made spot $n + 1$ a possible parking preference, we know that cars are now allowed to park here. Because the street is circular and there are more parking spaces than there are cars, all cars will eventually find an empty spot and park. Since there is one more parking space than there are cars, when all cars have parked, there will be one empty space after all cars have parked.

Connection to Example 1: Cars are allowed to park in spot 4. Since we have 3 cars and 4 spots, we know that each car will be able to park.

Sentence 4: "α is a parking function if and only if the empty space is $n + 1$."

This sentence restates our definition of a parking function. Consider $\alpha = (a_1, \ldots, a_n)$. This is new notation: α is an arbitrary preference vector, and each a_i for $i = 1, \ldots, n$ represents the preferred spot of the ith car. Earlier, we mentioned that α is a parking function if and only if it allows all of the cars to park on a street with n spots available. Thus, if we add an $n + 1$st spot at the end of the street, this new spot must be the empty spot if α is to be a parking function. If we have a car parked in the $n + 1$st spot of the circle, then that means that that car would not have been able to park on the (linear) street with n spots, and thus, α would not be a parking function.

Connection to Example 1: On the linear street, we had 3 cars and 3 spots. If we had a preference vector that allowed all of the cars to park, then it only utilized spots 1, 2, and 3, such as preference vector (1, 2, 2). Note that the preference vector (2, 3, 3) is not a parking function in the linear arrangement of the street because the third car would be unable to park. However, in the circular arrangement of the street, the preference vector (2, 3, 3) would allow the cars to park in spots 2, 3, and 4, respectively. But as we noted, this would not be a parking function because it would require the third car to park in the fourth spot which did not exist in the linear arrangement.

Sentence 5: "If $\alpha = (a_1, \ldots, a_n)$ leads to car c_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n + 1$) will lead to car c_i parking at space $p_i + j$ (modulo $n + 1$)."

The first part of this step defines our preference vector α in terms of its entries. That is, for $i = 1, \ldots, n$, we have that a_i represents where car c_i wants to park. Then, using α, we determine where car c_i actually parks, and call this spot p_i.

Now, we have to use some modular arithmetic. Let j be an integer such that $0 \leq j \leq n$. We could let j be $n + 1$, but we are working modulo $n + 1$, so $n + 1 = 0$ (mod $n + 1$), and 0 is a nicer number to work with. Note that this means we can also refer to spot $n + 1$ as spot 0.

The second part of this sentence explains the effect of shifting all the preferences by the same amount. We can break it down by understanding the effect of adding 1 (modulo $n + 1$) to all the preferences.

Lemma 3. Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a parking function for a circular street with $n + 1$ parking spaces in which cars park in the following order $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ where the p_i are all distinct. Define $\alpha + 1$ to be the vector $(a_1 + 1, a_2 + 1, \ldots, a_n + 1)$ (modulo $n + 1$). The preference vector $\alpha + 1$, results in the cars parking in order $\boldsymbol{p} = (p_1 + 1, p_2 + 1, \ldots, p_n + 1)$.

Proof. To prove this, we proceed by contradiction. Assume that i is the smallest index between 1 and n, inclusive, such that car c_j prefers spot $a_j + 1$ and parks in spot $p_j + 1$ for all $j < i$, while car c_i prefers spot $a_i + 1$ and parks in spot $q \neq p_i + 1$ (mod $n + 1$), where $1 \leq q \leq n + 1$. That is, car c_i prefers spot $a_i + 1$ and parks in a spot distinct from $p_i + 1$ (modulo $n + 1$). Let's begin by discussing the scenarios in which car c_i would be unable to park at spot $p_i + 1$ (modulo $n + 1$).

> Case 1: Another car took that spot! But if this is the case, then there exists some $j < i$ such that car c_j parks at spot $p_i + 1$ under the parking preference vector $\alpha + 1$. However, we know that car c_j parks at spot $p_j + 1$, by assumption. This implies that $p_i + 1 = p_j + 1$ (mod $n + 1$) which implies that $p_i = p_j$ (mod $n + 1$). Hence, $i = j$. However, we know that $i \neq j$ because α is a parking function, which means that no two cars can park in the same spot. Thus, we have a contradiction!
>
> Case 2: Spot $p_i + 1$ was available! Since car c_i does not park there, we know that when car c_i entered the circular one-way street and started going forward from its preferred spot $a_i + 1$ (modulo $n + 1$), it must have found an empty parking space before reaching spot $p_i + 1$ (modulo $n + 1$). By assumption, we know that cars c_1 through c_{i-1} were parked in spots $p_1 + 1$ through $p_{i-1} + 1$ (modulo $n + 1$) when car c_i entered the street. Since car c_i, starting at parking space $a_i + 1$ (modulo $n + 1$) encountered an empty spot (say parking space p) before reaching parking space $p_i + 1$ (modulo $n + 1$), we claim that had all the cars used parking function α instead of $\alpha + 1$, car c_i, starting at parking space a_i (modulo $n + 1$) would encounter an empty space before getting to space p_i, namely, the parking space $p - 1$ (modulo $n + 1$). We know this because by our assumption, if the cars use parking function α, all cars c_1 through c_{i-1} would be in spots p_1 through p_{i-1} (modulo $n + 1$), and if one of these spots was parking spot $p - 1$, then using parking function $\alpha + 1$, spot $(p - 1) + 1 = p$ (modulo $n + 1$) would be occupied. This gives rise to another contradiction!

Since we have handled all possible cases, the scenario where $q \neq p_i + 1$ (mod $n + 1$) does not happen. Therefore, $q = p_i + 1$ as desired. □

Using Lemma 3, we can iterate the process to establish that this result holds for all j (modulo $n + 1$). Namely, given a parking function $\alpha = (a_1, a_2, \dots, a_n)$ which results in cars parking in the order $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$, for any $1 \le j \le n + 1$, the parking preference vector $\alpha + j$, which we define to be $(a_1 + j, a_2 + j, \dots, a_n + j)$ (modulo $n + 1$), on the circular street with spots labeled 1 through $n + 1$, results in the cars parking in order $\boldsymbol{p} + j = (p_1 + j, p_2 + j, \dots, p_n + j)$ (modulo $n + 1$). This was the statement of sentence 5.

Connection to Example 1: Say, we considered preference vector (1, 2, 2). This would result in the following:

c_i	α_i	p_i
1	1	1
2	2	2
3	2	3

Now, we have to shift each a_i and p_i by j. Note that j can take all of its assigned integer values from 0 to n, which we consider individually (taken modulo $n + 1$) and present in the table below:

c_i	$a_i = a_i + 0$	$p_i = p_i + 0$	$a_i + 1$	$p_i + 1$	$a_i + 2$	$p_i + 2$	$a_i + 3$	$p_i + 3$
1	1	1	2	2	3	3	0	0
2	2	2	3	3	0	0	1	1
3	2	3	3	0	0	1	1	2

You should now verify that under the new parking preference vectors $\alpha + j$, the cars c_1, c_2, and c_3 actually park in spot $p_i + j$ (all considered modulo 4).

Sentence 6. "Hence, exactly one of the vectors $(a_1 + k, a_2 + k, \dots, a_n + k)$ (modulo $n + 1$) is a parking function, so $f(n) = (n + 1)^n/(n + 1) = (n + 1)^{n-1}$."

Now, we study each preference vector that exists for a particular value of n. So, for any given preference vector, we shift the corresponding a_i and p_i by j, both taken modulo $n + 1$. Studying a single preference vector and the resulting shifted preference vectors, we find that, of this set of $n + 1$ preference vectors, only one is a parking function. This is because all but one of them has a car c_i such that $p_i + j = 0 \pmod{n + 1}$ which is problematic because we mentioned earlier that we need the 0th parking spot (which is the same as the $n + 1$st parking spot) to be empty in order for α to be a parking function. Therefore, for each α, we need to take the one shifted preference vector $\alpha + k$ that has no p_i such that $p_i + k = 0 \pmod{n + 1}$, which will guarantee that the $n + 1$st spot is the empty spot.

To find the number of parking functions, we count them in groups. First, recall that there are $(n + 1)^n$ distinct possible preference vectors α.

Next, we group these preference vectors in the following way: If $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ are two preference vectors satisfying that $(a_1, \dots, a_n) = (b_1 + j, \dots, b_n + j) \pmod{n + 1}$, then α and β are in the same group. We next notice that each such group consists of $n + 1$ preference vectors as we can generate them from α by taking $\alpha + j$ for any value of j from 0 to n, inclusive. However, for each starting preference vector α, only one of these shifted vectors actually results in a parking function for the original linear street. So, we can pick this one parking function from each group of $n + 1$ preference vectors. Algebraically, we represent this by dividing the total number of preference vectors by $n + 1$. Therefore, we have finally established that the total number of parking functions is given by

$$|PF_n| = (n + 1)^n/(n + 1) = (n + 1)^{n-1}.$$

Connection to Example 1: Using our example table from Sentence 5 above, we can see that the column $p_i + 0$ is the only p_i column that does not contain a 0. That is, the column $a_i + 0$, corresponding to the preference vector (1, 2, 2), is the only preference vector that results in a parking function. Now, we know that for each of the 27 preference vectors that exist, each generates $3 + 1 = 4$ shifted vectors. Thus, we have $4^3 = 64$ vectors. By shifting, we are sorting these 64 shifted vectors into groups of $n + 1$ vectors, and only 1 of the vectors in each group works. Therefore, when we divide by $n + 1 = 4$, we find that there are $64/4 = 16$ working parking functions, which we discovered earlier in Example 1! Observe that 16 groups of 4 is a relatively large number of things to list out. We will work through two interesting examples together, and the remaining 14 will be for you to try out on your own.

Example 2. For the first group, we will consider the vector $\alpha = (1, 2, 2)$ and the vectors that it generates. Using this α, we can fill out our table below with each car's preferred parking spot and actual parking spot. (This is the same table we generated for Sentence 5.) We find

c_i	$a_i = a_i + 0$	$p_i = p_i + 0$	$a_i + 1$	$p_i + 1$	$a_i + 2$	$p_i + 2$	$a_i + 3$	$p_i + 3$
1	1	1	2	2	3	3	0	0
2	2	2	3	3	0	0	1	1
3	2	3	3	0	0	1	1	2

as the compilation of this information. In the $p_i + 0$ column, we see that no car parks in the 0th spot. In the $p_i + 1$ column, we see that car 3 parks in the 0th parking spot. In the $p_i + 2$ column, we see that car 2 parks in the 0th parking spot. In the $p_i + 3$ column, we see that car 1 parks in the 0th parking spot. Thus, we observe that only one column, namely $p_i + 0$, has no car parking in the 0th parking spot. From our work earlier, we know that this implies that the corresponding column of $a_i + 0$ is, indeed, a parking function because the 0th parking spot is vacant. Therefore, of this group of four preference vectors, only (1, 2, 2) is a parking function.

Example 3. For this group, we will consider the vector $\alpha = (2, 3, 2)$ and the vectors that it generates. Using this α, we can fill out our table below with each car's preferred parking spot and actual parking spot. We find

c_i	$a_i = a_i + 0$	$p_i = p_i + 0$	$a_i + 1$	$p_i + 1$	$a_i + 2$	$p_i + 2$	$a_i + 3$	$p_i + 3$
1	2	2	3	3	0	0	1	1
2	3	3	0	0	1	1	2	2
3	2	0	3	1	0	2	1	3

as the compilation of this information. In the $p_i + 0$ column, we see that car 3 parks in the 0th spot. In the $p_i + 1$ column, we see that car 2 parks in the 0th parking spot. In the $p_i + 2$ column, we see that car 1 parks in the 0th parking spot. In the $p_i + 3$ column, we see that no car parks in the 0th parking spot. Thus, we observe that only one column, namely $p_i + 3$, has no car parking in the 0th parking spot. From our work earlier, we know that this implies that the corresponding column of $a_i + 3$ is, indeed, a parking function because the 0th parking spot is vacant. Therefore, of this group of four preference vectors, only (1, 2, 1) is a parking function.

3. What's Next?

3.1 How Can I Generate a Research Question?

Wow, that was a lot of cool math! You might be curious about what else there is to know about parking functions, and we hope you will enjoy knowing that the study of parking functions and their generalizations fuels a very active area of mathematical research, and there are many questions still unanswered. You might be thinking "How do they come up with these questions?" The answer is: they just ask. We are sure you had a lot of questions as you read through this paper, and some of those would be great research questions if no one knows the answer yet. So, you should always ask questions. Especially those of the type "what happens if …?"

If you want to ask more about parking functions, there are two things you need to consider about your "parking world" in order to propose your very own research question.

1. *What do we know about the cars and parking spots?*

   By this, we mean that you should explicitly describe any information we have about the cars. How many cars are there? How many types of cars are there? Are there motorcycles, so two of them can park in the same spot? How many parking spots are there? Are any of them already occupied? In the example we studied here, we had the same number of cars and parking spots, all of the parking spots were available for parking, and all of our cars were identical.

2. *What is the parking rule?*

   By this, we mean that you should explicitly list any existing structures that govern how the cars in your scenario are able to park. In the example we studied here, the parking rule was that the cars could only drive one way, the cars would take their preferred spot if it was available, and if not, the cars would take the next available spot down the street. How would you adapt these rules in your own parking scenario?

3.2 What Are Mathematicians Currently Studying?

The table on the next page lists a few questions that mathematicians are actively researching! Most of the questions start out with counting how many preference vectors yield their version of parking functions. There are other things that mathematicians are currently studying, including "How do I generate all of the parking functions?" or "What is the probability that a car will get its preferred parking spot?" – these last are called lucky cars. A fun name indeed!

Name of Question	Brief Summary
Parking Completions [1]	This is a type of parking function where some of the parking spots are occupied or otherwise obstructed before we begin to allow the cars in question to start parking. Following the same rules as the parking functions in this paper, they enumerate the number of parking completions which are the parking preferences that allow the cars to park given the initial obstructions.
Interval Parking Functions [2]	These parking functions follow the same rules as the parking functions that we explored in this paper. However, the major difference is that each car not only has a preferred spot, which is listed in the preference vector, but each car also has a preference interval. So, as the cars are parking, if a car has an actual parking spot that is outside of its preference interval, then we do not have a parking function.
k-Naples Parking Functions [3, 4]	In the beginning of the paper, we talked about our one-way street having a dead end. That meant that if a car's preferred parking spot and all of the succeeding ones were occupied, the car couldn't park. You might have wondered, why can't the car back up? It turns out that with k-Naples parking functions, the car can reverse to see if there were any available parking spots prior to its preferred spot. These authors were studying what changes depending on how many spots the car can reverse to find an open spot.
k-Naples Parking Functions Statistics [5]	This paper does something that many researchers do by taking an existing scenario and asking a different question about it. These authors study the parking world that is k-Naples parking functions. However, instead of asking how many k-Naples parking functions there are, they study other information such as the order the cars actually park, and we call information of this type *statistics* about the parking functions.

Table 1. Some questions of current interest

References

[1] A. Adeniran, S. Butler, G. Dorpalen-Barry, P. E. Harris, C. Hettle, Q. Liang, J. L. Martin, and H. Nam. Enumerating parking completions using join and split. *The Electronic Journal of Combinatorics*, 27(2):#P2.44, 2020.

[2] E. Colaric, R. DeMuse, J. Martin, and M. Yin. Interval parking functions. *ArXiv*, 2020. https://arxiv.org/pdf/2006.09321.pdf.

[3] J. Carlson, A. Christensen, P. E. Harris, Z. Jones, and A. Ramos Rodriguez. Parking Functions: Choose Your Own Adventure, *ArXiv*, 2020. https://arxiv.org/pdf/2001.04817.pdf.

[4] A. Christensen, P. E. Harris, Z. Jones, M. Loving, A. Ramos Rodriguez, J. Rennie, and G. Rojas Kirby. A generalization of parking functions with backward movement. *The Electronic Journal of Combinatorics*, 27(1):#P1.33, 2020.

[5] L. Colmenarejo, P. E. Harris, Z. Jones, C. Keller, A. Ramos R., Eunice S., and A. R. Vindas-Meléndez. Counting k-naples parking functions through permutations and the k-naples area statistic, 2020. *ArXiv*, 2020. https://arxiv.org/pdf/2009.01124.pdf.

[6] N. Shales. Recursively counting a parking function, retrieved October 10, 2020. https://math.stackexchange.com/questions/2718303/recursively-counting-a-parking-function.

[7] R. P. Stanley. Parking functions, 2018, retrieved October 8, 2020. http://www-math.mit.edu/rstan/transparencies/parking.pdf.

Valentine's Math, Part 1

by Ken Fan | edited by Jennifer Sidney Silva

Emily: Jasmine, check out this Valentine heart app![1]

Jasmine: That's neat!

Emily: And look, it shows the mathematical formula that's being graphed. You can change the parameters and vary the shape of the Valentine heart.

Emily adjusts one of the parameters as they watch the dimple at the top of the heart sharpen.

Jasmine: That's a complicated equation!

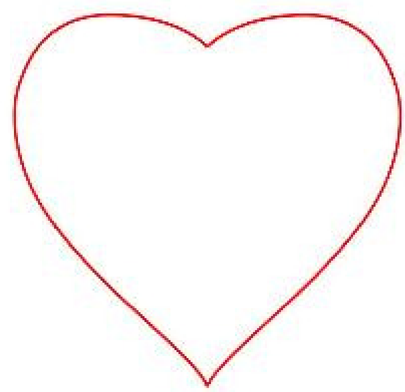
Emily: Well, the graph of this equation is a surface in 3D, so it involves x, y, and z.

Jasmine: I never thought to try to find a mathematical equation that describes a Valentine heart. Maybe we can come up with one for a 2D Valentine heart. Do you want to try?

Emily: In other words, you want to find some equation involving the variables x and y for which the solutions (x, y), when plotted, form the boundary of a Valentine heart?

Jasmine: Exactly. Just as the solutions to the equation $x^2 + y^2 = 1$, when plotted, yield a circle, let's find an equation whose solutions, when plotted, form the boundary of a Valentine heart.

Emily: Sure, sounds like fun! I guess we can start with a sketch of a Valentine's heart.



Emily draws the heart shape shown at right.

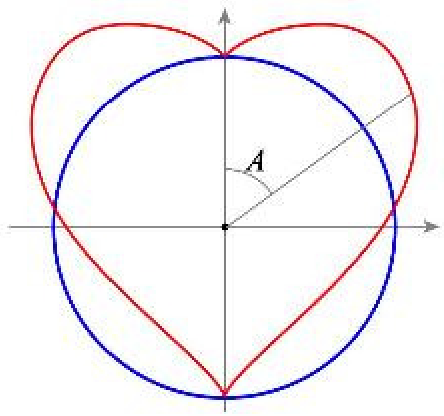
Jasmine: That's nicely drawn, Emily! But I have no idea what equation could describe such a shape. I guess it's like a circle that's been pinched out at the bottom and has a dimple on top. Maybe we can somehow modify the equation that describes a circle, to produce those features.

Emily: Since all circles are similar to each other, we might as well use the circle described by the equation $x^2 + y^2 = 1$. But how can we modify it to produce, say, the dimple on top?

Jasmine: Let's see. The equation $x^2 + y^2 = 1$ says that the square of the distance of the point (x, y) from the origin is equal to 1. If we place our heart in the coordinate plane so that it contains the origin and is symmetric about the y-axis, then as we travel around the heart, our distance from the origin will vary. Perhaps we could replace the 1 on the right-hand side of the equation with the square of a formula that describes how this distance varies.

[1] Emily is looking at the widget at https://love.imaginary.org/ created by Aaron Montag.

Emily: That sounds like a good plan! Let's place the origin of coordinates exactly halfway between the bottom of the dimple and the bottom of the pinch. I'll add the axes as well as a circle so we can more easily guesstimate the distances of points on the heart from the origin. And we might as well scale it so that the circle is a unit circle.



Jasmine: If we can figure out the distances from the origin to various points on the heart as a function of the angle A, I think we'll essentially have our equation. All we'd have to do is replace the angle A with a function that gives A in terms of x and y, then replace the 1 in $x^2 + y^2 = 1$ with the square of the result.

Emily: Let's sketch a graph of the distances of points on the heart from the origin as a function of the angle A, and let's refer to this function as the "distance function."

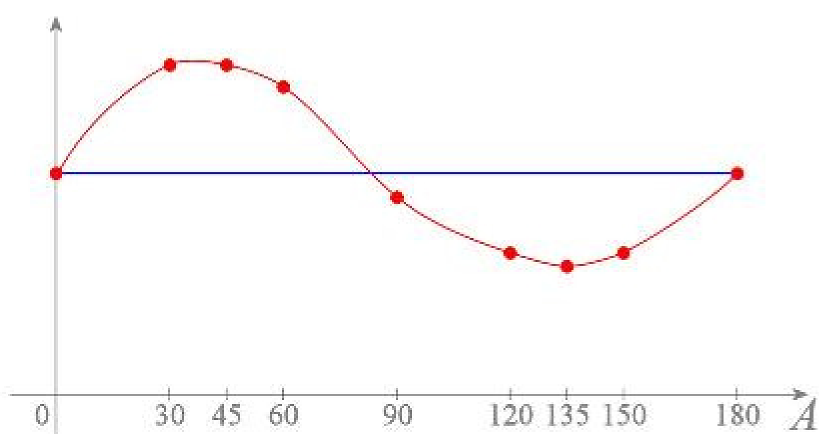
Jasmine: Okay. When A is 0°, we're at the bottom of the dimple; so the distance will be the radius of the circle, which we're taking to be 1. And the distance will also be 1 when A is 180°, by construction. When A is 90°, that's the x-intercept of the right lobe of the heart, and I'd say that the distance is about 7/8?

Emily: That seems about right.

Emily and Jasmine make the following table:

A	0°	30°	45°	60°	90°	120°	135°	150°	180°
Distance	1	3/2	3/2	11/8	7/8	2/3	5/8	2/3	1

Emily: If we sketch out an extrapolation of these values, we get something that looks more or less like this.



Jasmine: There's no definitive Valentine heart shape, so we have some leeway. But the graph looks roughly like a sine wave. What if we just use a sine wave? What would the resulting heart look like?

Emily: I don't know, let's try it and see!

Jasmine: Our graph looks something like the graph of $1 + \sin(2A)/2$, so let's use that for our distance function.

Emily: Okay. We also have to figure out how to relate A to (x, y). For points in the first quadrant, where x and y are both positive, the angle with measure A is one of the acute angles of a right triangle with legs of length x and y; so if (x, y) is in the first quadrant, then A is the arctangent of x/y. In the second quadrant, where x is positive and y is negative, the arctangent of x/y will be something between -90° and 0°, so I think that $A = 180° + \arctan(x/y)$.

Jasmine: For quadrants three and four, we're actually measuring A counterclockwise from the positive y-axis. So I think we can use the same formulas you got for the first and second quadrants, except that we have to take the absolute value of x. In other words, for $y \geq 0$, we can use the formula $A = \arctan(|x|/y)$, and for $y < 0$, we can use the formula $180° + \arctan(|x|/y)$.

Emily: It bothers me a little that we have to split into cases, depending on whether y is positive or negative. After all, as we travel around the boundary of the heart from the point (0, 1) to the point (0, -1), the angle changes continuously from 0° to 180°.

Jasmine: Maybe we can combine them into a single formula, because the basic arctangent function goes continuously from -90° at negative infinity to 90° at positive infinity. Typically, angles are measured from the positive x-axis, and not the positive y-axis, so if we revert to the typical way of realizing the arctangent, I think that we can get A with the formula

$$A = 90° - \arctan(y/|x|).$$

In her head, Emily thinks it through: *when $x > 0$, we have $\arctan 1/x = 90° - \arctan x$. So for $y > 0$, we have $90° - \arctan(y/|x|) = 90° - (90° - \arctan(|x|/y)) = \arctan(|x|/y)$, which agrees with our first formula. And for $x < 0$, we have $\arctan 1/x = -90° - \arctan x$. So for $y < 0$, the formula $90° - \arctan(y/|x|)$ is equal to $90° - (-90° - \arctan(|x|/y)) = 180° + \arctan(|x|/y)$, which also agrees with our formula!*

Emily: I agree! So our Valentine heart equation is

$$x^2 + y^2 = (1 + \sin(2(90° - \arctan(y/|x|))/2)^2. \qquad (1)$$

Jasmine: Let's simplify that.

Emily and Jasmine simplify the equation, using the trigonometric identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

and

$$\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

For details, see the box at right.

Emily: I got

$$x^2 + y^2 = \left(1 + \frac{y|x|}{x^2+y^2}\right)^2.$$

Simplification of Equation 1

$$x^2 + y^2 = (1 + \sin(2(90° - \arctan(y/|x|))/2)^2$$

$$= (1 + \sin(180° - 2\arctan(y/|x|))/2)^2$$

$$= (1 + \sin(2\arctan(y/|x|))/2)^2$$

$$= (1 + \sin(\arctan(y/|x|)) \cos(\arctan(y/|x|))^2$$

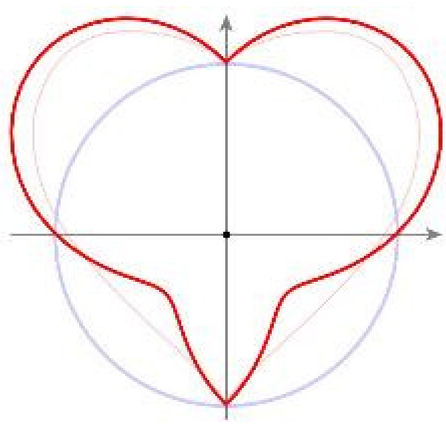
$$= \left(1 + \frac{y}{\sqrt{x^2+y^2}} \frac{|x|}{\sqrt{x^2+y^2}}\right)^2$$

$$= \left(1 + \frac{y|x|}{x^2+y^2}\right)^2.$$

Perhaps it can be simplified further by expanding, but I like the way the form of the expression exposes our original idea.

Jasmine: Let's graph it!

Emily and Jasmine use a computer to plot the graph of their equation. Here's the output superimposed over the unit circle (in blue) and their original heart sketch (in faded red):



Emily and Jasmine let out a laugh.

Jasmine: Yikes! That does *not* look like a Valentine heart!

Emily: It kind of looks like a stylized ram's head.

Jasmine: A ram's head?

Emily: I don't know. Actually, it's not that far off from our original sketch. I think it represents progress. It looks like the perturbation from the circle is just a little too much all around. Perhaps we should use a smaller amplitude for the sine curve.

Jasmine: Gosh, I wish we used a variable for that amplitude so we wouldn't have to redo the derivation! Let's redo it. But this time, for the distance function, let's use $1 + a\sin(2A)$.

Emily and Jasmine redo their derivation using a variable for the amplitude of the sine wave.

Jasmine: Okay, I get

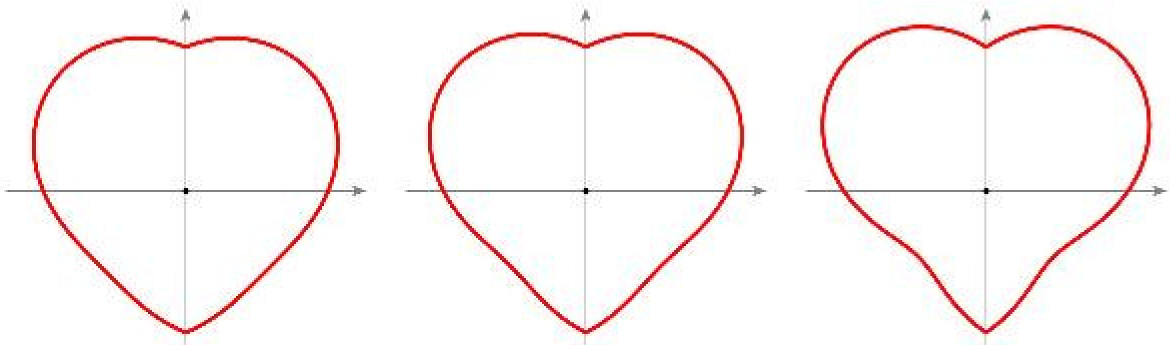
$$x^2 + y^2 = \left(1 + \frac{2ay|x|}{x^2 + y^2}\right)^2.$$

Emily: I get that, too. What value of a would you like to try?

Jasmine: I'm not sure. I guess we should try several values of a. We've already seen the graph for $a = 1/2$, so why don't we try the values 1/5, 1/4, and 1/3?

Emily: Sounds good to me.

Emily and Jasmine write a quick computer program to plot the equation. Here's the output:



The graphs of $x^2 + y^2 = \left(1 + \frac{2ay|x|}{x^2 + y^2}\right)^2$ with a = 1/5, 1/4, and 1/3, respectively, from left to right.

Jasmine: The graph with a = 1/5 looks most like a Valentine heart to me, but all of them bulge at the bottom instead of tapering there.

Emily: For the top halves, I actually prefer the larger values of a.

Jasmine: I do, too.

Emily: Perhaps we need to abandon the idea of using a simple sine function for our distance function. Let's try to refine our distance function to look like 1 + sin(2A)/**2** for angles A corresponding to the upper part of the heart, and more like 1 + sin(2A)/**5** for angles A corresponding to the lower part of the heart. It would be nice if we could also make the heart taper to a point at the bottom instead of bulging the way it does in all the examples we've constructed so far.

Jasmine: Yes, but how do we make it taper instead of bulge?

Emily: At the very bottom of the heart, where the angle A is 180°, the distance function returns 1. As the angle A decreases, we need that distance function to drop below 1 faster.

Jasmine: If it drops off faster, couldn't that still result in a bulge? For example, when our amplitude a is bigger, the distance function drops from 1 faster, but we still get a bulge instead of a taper.

Emily: You're right. It's got to drop off in a way that makes the curve bend away from the y-axis, instead of toward it. In any case, if we are much more careful about the exact values of the distance function for our heart sketch, then build a function that accurately models that, we should get a heart that tapers at the bottom.

Jasmine: Hmm. I bet it'll be hard to find an explicit function that accurately models the distance function associated with our heart sketch. But at the moment, I don't have a better idea to try.

Can you come up with an equation that produces a nicer-looking Valentine heart? If you do, please share it with us at girlsangle@gmail.com.

To be continued...

Notation

by Girls' Angle Staff
edited by Amanda Galtman

Good notation facilitates communication. To learn notation, use it. Practice makes perfect!

Matrices and Linear Equations

An important topic from Algebra I is solving systems of linear equations, like this one:

$$2x + 3y = 7$$

$$4x - 5y = 10$$

Here, we won't discuss why these are important or how to find solutions. Instead, we'll focus on one notation commonly used to express such systems, called **matrices**.

Matrix notation expresses the above system of equations like this:

$$\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}.$$

Let's take the original system and let everything fade out, except the numbers:

$$\begin{array}{c} 2x + 3y = 7 \\ 4x - 5y = 10 \end{array} \quad \rightarrow \quad \begin{array}{c} 2x + 3y = 7 \\ 4x - 5y = 10 \end{array} \quad \rightarrow \quad \begin{array}{ccc} 2 & +3 & 7 \\ 4 & -5 & 10 \end{array}$$

We end up with an array of numbers. Why can't we just use this unadorned array of numbers to express our system of linear equations?

Well, for one thing, we've lost the variables. For another, how would we know if this 2 by 3 array of numbers corresponds to the intended system, or to the system

$$2 = 3x + 7y$$

$$4 = -5x + 10y,$$

or to something else that has the same numbers used in different ways?

Matrix notation is designed to allow us to specify a system of linear equations precisely.

A matrix is a rectangular array of numbers set off from the surrounding material by a pair of parentheses (or square brackets). The matrix form of our system actually consists of three matrices. In the context of linear equations, the two matrices on the left of the equation can be combined using an operation called **matrix multiplication** into a single matrix.

$$\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 4x-5y \end{pmatrix}.$$

Two matrices are equal to each other if they have the same dimensions and corresponding entries are equal. So our matrix equation, which equates two matrices that each have two rows and one column, corresponds to two equations, one for each position in the matrix. These two equations are the two equations in our original system of linear equations.

In fact, we can turn the desire to have $\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ represent $\begin{pmatrix} 2x+3y \\ 4x-5y \end{pmatrix}$ into the definition for matrix multiplication! To see matrix multiplication more clearly, we can replace the coefficients with variables:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

Here's a system of three linear equations in three unknowns *x*, *y*, and *z*, written using matrices and written as a standard system of linear equations:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \qquad \begin{aligned} x + 2y + 3z &= 10 \\ 4x + 5y + 6z &= 11 \\ 7x + 8y + 9z &= 12 \end{aligned}$$

We extend matrix multiplication so that the second factor can have more than one column by simply multiplying the first matrix with each column of the second and putting each result in a separate, corresponding, column in the result. We can also extend matrix multiplication in a similar way to include products where the first factor has a different number of rows. All that's needed to ensure that two matrices can be multiplied together is that the number of columns in the first factor is the same as the number of rows in the second factor. Here's an example:

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix}\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} = \begin{pmatrix} au+bx & av+by & aw+bz \\ cu+dx & cv+dy & cw+dz \\ eu+fx & ev+fy & ew+fz \\ gu+hx & gv+hy & gw+hz \end{pmatrix}.$$

The product has as many rows as the first factor and as many columns as the second factor.

Note that we haven't actually defined matrix multiplication in general. Based on the examples, how would you define it?

There are many ways to interpret matrices and many ways to think about matrix multiplication, and there are applications galore! If you are interested in exploring further, see if you can show that matrix multiplication is associative, but not commutative. Also, check out any book on linear algebra.

Notes from the Club

These notes cover some of what happened at Girls' Angle meets. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material. Less than 5% of what happens at the club is revealed here.

Session 28 - Meet 1 January 28, 2021	Mentors:	Cecilia Esterman, Kimberly Hadaway, Adeline Hillier, Bridget Li, Tingyi Lu, Kate Pearce, AnaMaria Perez, Nehar Poddar, Gisela Redondo, Vievie Romanelli, Savannah Tynan, Emma Wang, Jane Wang, Karissa Wenger, Rebecca Whitman, Annie Yun

Traditional 3×3 tic-tac-toe always ends in a draw if each player plays optimally, but what should happen if you play $3 \times 3 \times 3$ tic-tac-toe?

Session 28 - Meet 2 February 4, 2021	Mentors:	Cecilia Esterman, Kimberly Hadaway, Adeline Hillier, Bridget Li, Kate Pearce, AnaMaria Perez, Nehar Poddar, Gisela Redondo, Vievie Romanelli, Savannah Tynan, Emma Wang, Jane Wang, Karissa Wenger, Rebecca Whitman, Annie Yun

Several of our members love cats, so naturally, they dreamt up math problems involving cats. For example: What is the optimal size and shape for a cat carrier? What is the probability distribution of paw steps on a stairs as the cat runs up and down one?

Can you think of any math questions involving your pets?

Session 28 - Meet 3 February 11, 2021	Mentors:	Cecilia Esterman, Kimberly Hadaway, Adeline Hillier, Bridget Li, Kate Pearce, Nehar Poddar, Vievie Romanelli, Savannah Tynan, Emma Wang, Jane Wang, Rebecca Whitman, Angelina Zhang, Rachel Zheng

In the previous issue, Bogosian, Catanzaro, Cavatorta, and Danison introduced and analyzed the Toblerone game. It inspired another group to make up their own game to analyze. How would you want to modify the Toblerone game? What are the characteristics of a good game? Can you invent a game that has all the characteristics you want in a good game?

Session 28 - Meet 4 February 25, 2021	Mentors:	Talia Blum, Cecilia Esterman, Kimberly Hadaway, Adeline Hillier, Rebecca Nelson, Kate Pearce, AnaMaria Perez, Nehar Poddar, Gisela Redondo, Vievie Romanelli, Karissa Wenger, Rebecca Whitman, Rachel Zheng

A lot of geometric questions arose this session, such as: What shape maximizes area for a given perimeter? What shape maximizes volume for a given surface area? How can you efficiently pack cylinders into a rectangular box? How does scaling affect area and volume?

Do these questions make you think of any questions?

Calendar

Session 27: (all dates in 2020)

September	10	Start of the twenty-seventh session!
	17	
	24	
October	1	
	8	
	15	
	22	
	29	
November	5	
	12	
	19	
	26	Thanksgiving - No meet
December	3	

Session 28: (all dates in 2021)

January	28	Start of the twenty-eighth session!
February	4	
	11	
	18	No meet
	25	
March	4	Daina Taimina, Cornell University
	11	
	18	
	25	No meet
April	1	
	8	
	15	
	22	No meet
	29	
May	6	

Girls' Angle has run over 150 Math Collaborations. Math Collaborations are fun, fully collaborative, math events that can be adapted to a variety of group sizes and skill levels. We now have versions where all can participate remotely. We have now run four such "all-virtual" Math Collaboration. If interested, contact us at girlsangle@gmail.com. For more information and testimonials, please visit www.girlsangle.org/page/math_collaborations.html.

Girls' Angle can offer custom math classes over the internet for small groups on a wide range of topics. Please inquire for pricing and possibilities. Email: girlsangle@gmail.com.

Girls’ Angle: A Math Club for Girls

Membership Application

Note: If you plan to attend the club, you only need to fill out the Club Enrollment Form because all the information here is also on that form.

Applicant’s Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address (the Bulletin will be sent to this address):

Email:

Home Phone: ____________________________ Cell Phone: ____________________________

Personal Statement (optional, but strongly encouraged!): Please tell us about your relationship to mathematics. If you don’t like math, what don’t you like? If you love math, what do you love? What would you like to get out of a Girls’ Angle Membership?

The $50 rate is for US postal addresses only. **For international rates, contact us before applying.**

Please check all that apply:

- ☐ Enclosed is a check for $50 for a 1-year Girls’ Angle Membership.
- ☐ I am making a tax free donation.

Please make check payable to: **Girls’ Angle**. Mail to: Girls’ Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com.

Girls' Angle

A Math Club for Girls

Girls' Angle
Club Enrollment

Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!

The club is where our in-person mentoring takes place. At the club, girls work directly with our mentors and members of our Support Network. To join, please fill out and return the Club Enrollment form. Girls' Angle Members receive a significant discount on club attendance fees.

Who are the Girls' Angle mentors? Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

What is the Girls' Angle Support Network? The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society.

What is Community Outreach? Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

Who can join? Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-12. We welcome *all girls* (in grades 5-12) regardless of perceived mathematical ability. There is no entrance test. Whether you love math or suffer from math anxiety, math is worth studying.

How do I enroll? You can enroll by filling out and returning the Club Enrollment form.

How do I pay? The cost is $20/meet for members and $30/meet for nonmembers. Members get an additional 10% discount if they pay in advance for all 12 meets in a session. Girls are welcome to join at any time. The program is individually focused, so the concept of "catching up with the group" doesn't apply.

Where is Girls' Angle located? Girls' Angle is based in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

When are the club hours? Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at www.girlsangle.org/page/calendar.html or send us email.

Can you describe what the activities at the club will be like? Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

Are donations to Girls' Angle tax deductible? Yes, Girls' Angle is a 501(c)(3). As a nonprofit, we rely on public support. Join us in the effort to improve math education! Please make your donation out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

Who is the Girls' Angle director? Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was a Benjamin Peirce assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study, and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science, worked in the mathematics educational publishing industry, and taught at HCSSiM. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum.

Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities? Girls' Angle has a stellar Board of Advisors. They are:

At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics? We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known. In this way, we hope to help girls become solvers of the yet unsolved.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

Girls' Angle: Club Enrollment Form

Applicant's Name: (last) ______________________________ (first) ______________________________

Parents/Guardians: ____

Address: ____ Zip Code: _________

Home Phone: _________________ Cell Phone: _________________ Email: ______________________

Please fill out the information in this box.

Emergency contact name and number: ____

Pick Up Info: For safety reasons, only the following people will be allowed to pick up your daughter. Names:

____

Medical Information: Are there any medical issues or conditions, such as allergies, that you'd like us to know about?

____

Photography Release: Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

Eligibility: Girls roughly in grades 5-12 are welcome. Although we will work hard to include every girl and to communicate with you any issues that may arise, Girls' Angle reserves the discretion to dismiss any girl whose actions are disruptive to club activities.

Personal Statement (optional, but strongly encouraged!): We encourage the participant to fill out the optional personal statement on the next page.

Permission: I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

____ Date: _______________________

(Parent/Guardian Signature)

Participant Signature: ____

Members: Please choose one.	Nonmembers: Please choose one.
☐ Enclosed is \$216 for one session (12 meets)	☐ I will pay on a per meet basis at \$30/meet.
☐ I will pay on a per meet basis at \$20/meet.	☐ I'm including \$50 to become a member, and I have selected an item from the left.

☐ I am making a tax free donation.

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to girlsangle@gmail.com. Also, please sign and return the Liability Waiver or bring it with you to the first meet.

Personal Statement (optional, but strongly encouraged!): This is for the club participant only. How would you describe your relationship to mathematics? What would you like to get out of your Girls' Angle club experience? If you don't like math, please tell us why. If you love math, please tell us what you love about it. If you need more space, please attach another sheet.

Girls' Angle: A Math Club for Girls

Liability Waiver

I, the undersigned parent or guardian of the following minor(s)

____,

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: ____Date: ________________

Print name of applicant/parent: ____

Print name(s) of child(ren) in program: ____________________________________

