

# Girls' *Angle* Bulletin

October 2009 • Volume 3 • Number 1

*To Foster and Nurture Girls' Interest in Mathematics*

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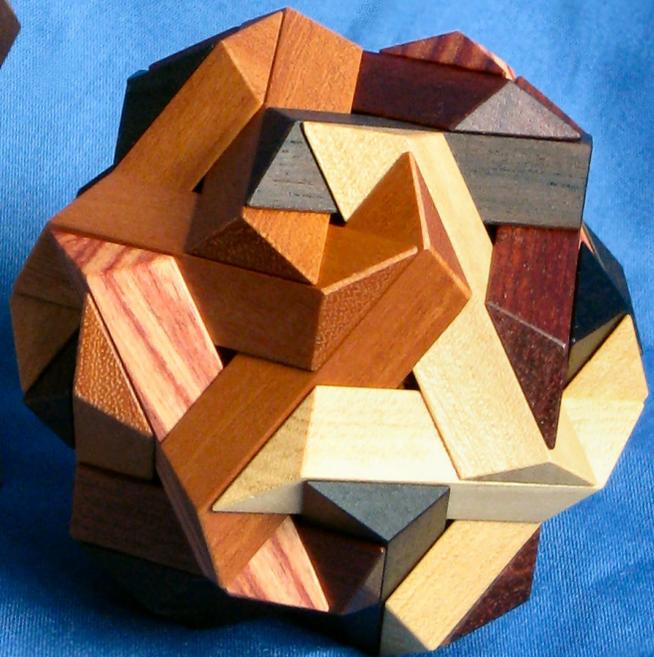
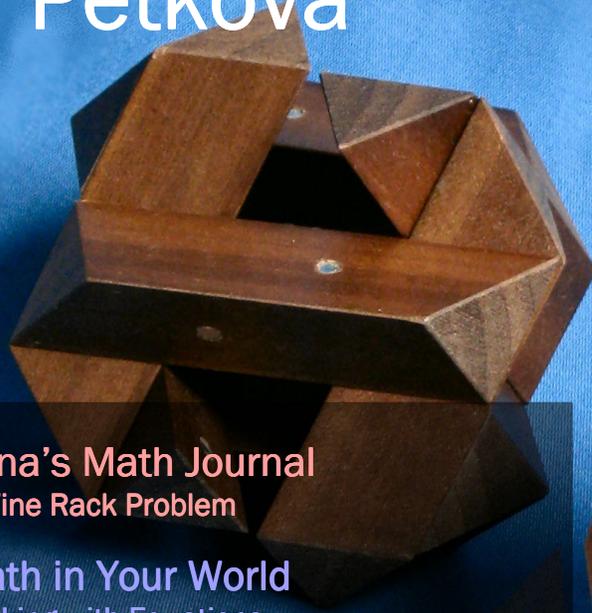
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# From the Director

New beginnings!

Club members will now receive a printed version of the Bulletin. We'll continue to make an electronic version available on our website for the near future.

And in these pages, read our first article by a club member, meet our first Women in Mathematics video presenter, **Ina Petkova**, and follow the *Adventures of Emmy Newton* in **Maria Monks'** new serial. If you haven't seen Ina's presentation of a proof of the Pythagorean theorem, it's on our website!

And finally, I'm bordering on ecstatic to announce our first tenured mathematics professor on the Advisory Board! **Gigliola Staffilani** is the Abby Rockefeller Mauzé Professor of Mathematics at MIT. Many of the girls met her last April.

Please enjoy these firsts as well as a second article by **Doris Dobi** on the Euclidean algorithm, a third by **Allison Henrich** on knots, and the beginning of a second year for our first regular columnists **Anna Boatwright** and **Katy Bold**!

All my best,  
Ken Fan  
Founder and Director

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## Girls' Angle Bulletin

*The official magazine of  
Girls' Angle: A Math Club for girls  
Electronic Version (ISSN 2151-5743)*

girlsangle@gmail.com

This magazine is published about six times a year by Girls' Angle to communicate with its members and to share ideas and information about mathematics.

Girls' Angle welcomes submissions that pertain to mathematics.

Editor: C. Kenneth Fan

## Girls' Angle: A Math Club for Girls

*The mission of Girls' Angle is to foster and nurture girls' interest in mathematics and empower them to tackle any field no matter the level of mathematical sophistication.*

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On the cover: Wooden polyhedral puzzles built by Jane Kostick.

# Much Ado About Knotting!

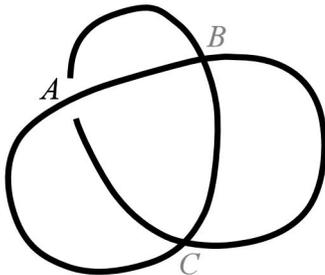
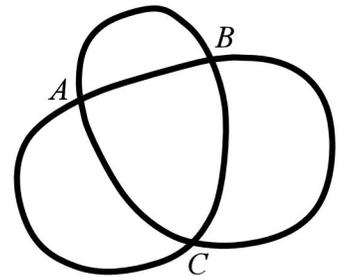
by Allison Henrich

In the past several months, we've learned a thing or two about knots. We discussed knotty concepts like Reidemeister moves and tricoloring and have made the acquaintance of knots such as the unknot and the trefoil. Now that we know some ways to tell if knot diagrams represent the same or different knots, it's a perfect time to use our knowledge to play some games! Without further ado, I give you...

## To Knot or Not to Knot

To illustrate how this game works, we'll sit in on a game that Ursula and Kate are playing. Ursula and Kate are given a knot diagram where all of the information at the crossings is missing. (This kind of diagram is often called the "shadow" of a knot.) The two take turns choosing over/under information at crossings, each with a different goal in mind. Ursula would like to make the diagram into a picture of the unknot, while Kate would like to make the diagram into a picture of a non-trivial knot (that is, ANYTHING BUT the unknot). Once either of them has made the diagram knotted or unknotted, the game is over.

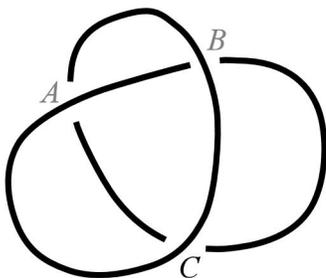
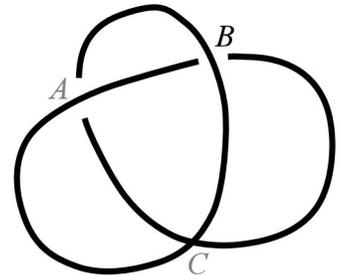
Let's say they start their first game with the shadow at right.



Suppose Ursula is allowed to go first and decides to make the move shown at left.

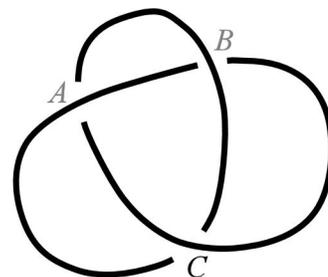
Using what you have learned about knots, check that there is a way of choosing the rest of the crossings to make the unknot, as well as a way to choose them that makes the diagram knotted. Thus, the game continues!

Now Kate wants to make the diagram knotted, so she chooses crossing *B* as shown at right.



What is interesting about Kate's move is that, if she had chosen the crossing *B* in the opposite way, it wouldn't matter what happened to crossing *C*. The knot would be unknotted! (Can you see why?) Kate was clever and avoided this catastrophe by choosing *B* the way she did. Unfortunately for Kate, however, Ursula still has the opportunity to win. Ursula chooses to unknot the knot by making the following move. Using our old friends, the Reidemeister moves, show that the knot at left is indeed the unknot.

Furthermore, notice that the knot would be the trefoil knot had Ursula made a different final move. It would look like the diagram at right. Recall that last time, we proved that the trefoil is different from the unknot using tricoloring.

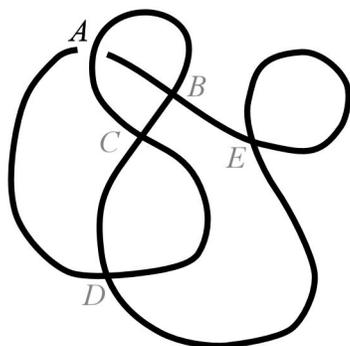
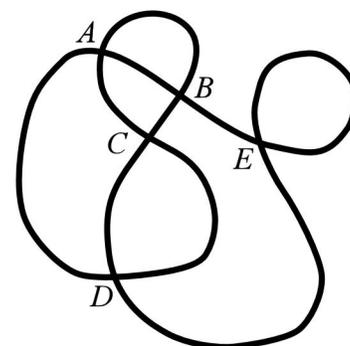


Now it seems that Ursula had the upper hand in this game, being allowed to play first. Could Kate have won if she had played first? Try playing the same game with a friend, allowing the knotting player to go first. Come back when you're done for a more complicated game!



Now that you see how the game works, why don't we watch the much-awaited rematch between our clever and knotty girls!

At right is Ursula and Kate's new starting knot shadow.

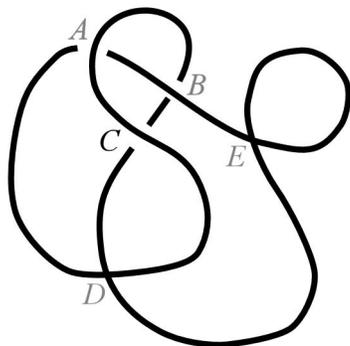
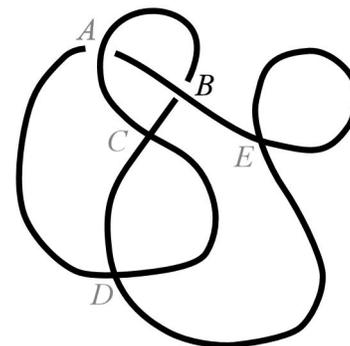


Let's say that Ursula, who is trying to unknot the knot, is chosen once again to go first. She makes the play at crossing A shown at left.

Notice that, while the type of knot we have is not yet determined, there is a way to choose the information at crossing B so that, no matter what happens with the other crossings, the knot will be unknotted. (Can you prove this? Here's the rough idea: Consider the loop that starts at crossing C and contains crossings A and B. Think about what would happen if B were chosen so that this loop passed entirely over the other strand of the knot. Then you could use a

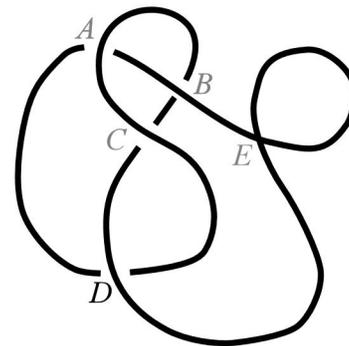
Reidemeister 2 move to remove crossings A and B. From there, Reidemeister 1 makes everything unravel no matter how you choose the remaining crossings!)

To avoid this disastrous outcome, Kate, who is trying to knot the knot, will decide crossing B so that it is still undetermined whether the diagram is knotted or not.



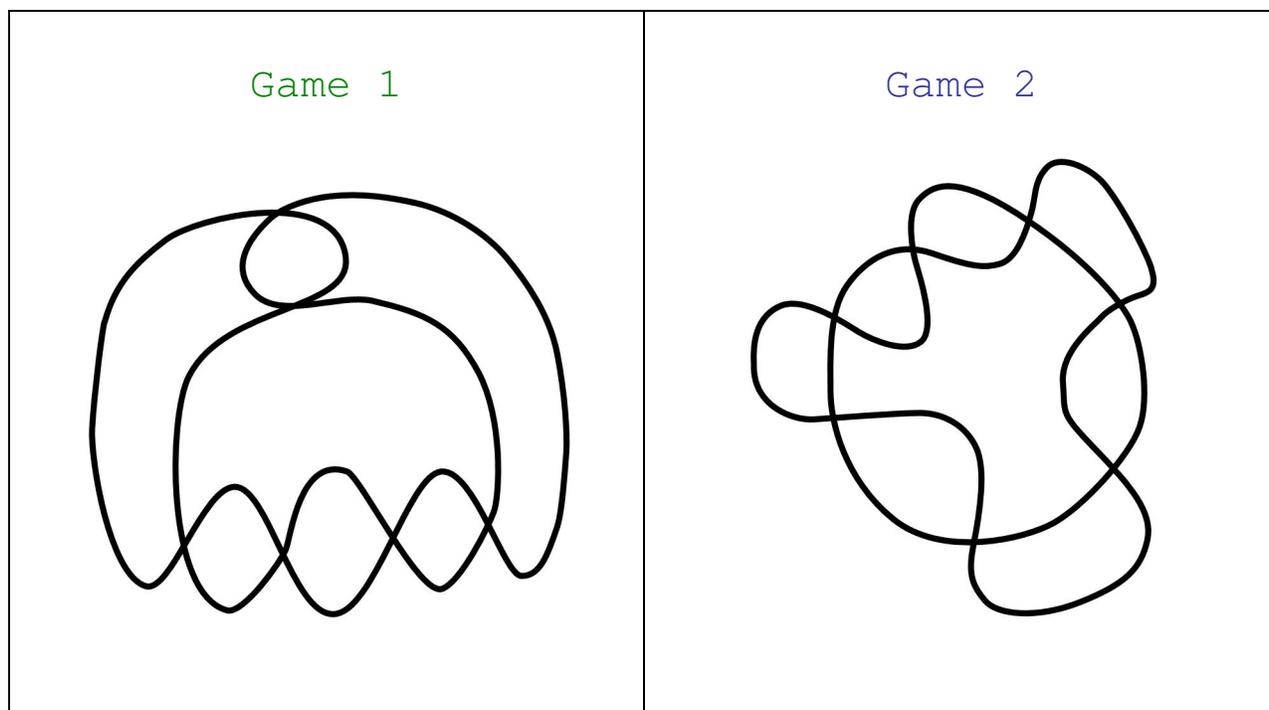
Now it is Ursula's turn. Where should she go? Is there any crossing she could determine so that the knot is unknotted? Think of all the possibilities! Let's say that she decides in the end to resolve crossing C in the way shown at left.

Kate can see now that determining crossing *D* one way will produce the unknot while determining it the other way will yield the trefoil! Thus, Kate chooses to do the move shown at right to win the game.



Let's think about why this move won the game for Kate. Note that, although crossing *E* is still undetermined, the game is over. Why? Can you show that either way we could determine crossing *E* will actually yield the same knot, and, furthermore, can you show that the resulting knot diagram can be related by Reidemeister moves to the standard diagram of the trefoil?

Now that you've had a crash course in knot games, are you ready to play your own game with a friend? Try taking turns being the player who knots and the player who unknots the diagram. Also, take turns going first on the same shadow. You can either play on the following shadows or create your own shadows to play on!



Once you've played several games, try answering the following questions.

1. For any particular shadow, is there a relationship between who goes first and who wins?
2. Does the player who knots ever have an advantage over the player who unknots or vice versa?
3. Given any of the shadows you've played on, if you are allowed to choose whether to go first or second and whether to knot or unknot, can you guarantee you'll win?

Have fun playing games!

# An Interview with Ina Petkova

Ina is a graduate student in mathematics at Columbia University. She presented a proof of the Pythagorean theorem in Girls' Angle's first Women in Mathematics video, now available for viewing on our website [www.girlsangle.org](http://www.girlsangle.org). Ina was born in Bulgaria.

**Ken:** Hi Ina, last month, we posted our first video in Eli's<sup>1</sup> series of videos featuring women in mathematics. It shows you explaining a proof of the Pythagorean theorem. Thank you for doing that video and for giving us this time to get to know you a little better. I'd like to start by giving you a chance to introduce yourself. Who is Ina Petkova?

**Ina:** I am a grad student at Columbia University. I study knot theory, and I am obsessed with playing hockey.

**Ken:** Hockey!? That's interesting...do you mean ice hockey or street hockey?

**Ina:** Ice hockey.

**Ken:** This is the first time I've ever interviewed a hockey player. What got you interested in mathematics? How old were you when you realized that you wanted to become a mathematician?

**Ina:** I feel like I was always interested in math, I'm not even sure how it started. I loved solving puzzles as I was growing up, and would constantly beg my grandfather to come up with new math riddles. The interest naturally evolved into taking more math classes in school, going to competitions, etc.

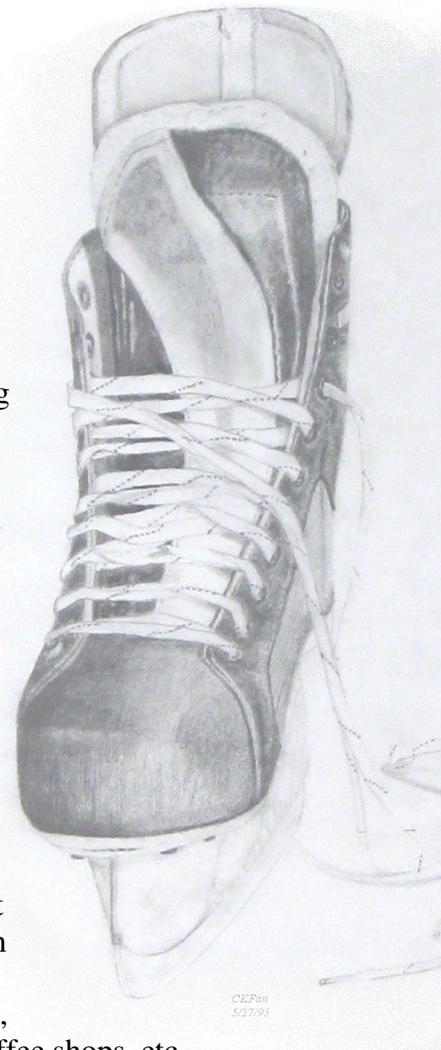
**Ken:** Do you find mathematics easy?

**Ina:** I find mathematics fun. The harder it gets and the more there is that you don't understand, the more fun it is, because that's when you get to ask others questions, look at more books, and end up learning more.

**Ken:** What is your daily life like as a graduate student in mathematics?

**Ina:** I don't really have a daily schedule, but I try to go to my office every day. I usually don't get there until noon, which is fine, because I think much better in the afternoon and in the evening. I'm lucky to have a lot of people at Columbia working in my field, so a lot of our time is spent just talking to each other about the problems we're working on, or the papers we're reading. I try to hit the gym at least a few times a week - in addition to keeping me in shape, it helps clear out the mind. There's a lot of coffee involved, hanging out in coffee shops, etc.

**Ken:** It seems like it takes a lot of self-discipline and motivation to succeed as a graduate student in mathematics. Do you have any advice for how best to learn mathematics?



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<sup>1</sup> Eli Grigsby is a Girls' Angle director and an assistant professor of mathematics at Boston College.

**Ina:** It does take a lot of self-discipline to be a grad student. One problem is that if you take a break for more than a few days, it's very hard to get back into working, especially if you're not working on a really exciting problem at the moment. It's sort of the same as playing an instrument - you have to do it every day to stay sharp. It's hard to say how one can best learn mathematics. Everyone has a different approach - some like reading books on their own, others prefer taking classes and learning from a teacher, some like reading more theory, and others like solving problems. The advice I can give to everyone is, don't be afraid to ask questions, communicate with others about what you're learning, and every time you learn something new, try to think of concrete examples.

**Ken:** I'm curious, why did you choose to present a proof of the Pythagorean theorem for our video series? Do you use the Pythagorean theorem in your work?

**Ina:** I chose that proof because it's famous, easy to understand, and very geometric. Often people think that math is only about solving equations, and it's neat to know that you can also prove things by drawing the right picture, for example. The Pythagorean theorem comes up in mathematics very often. One example is, this is how we can compute the distance between two given points in space.

**Ken:** What is one of the most memorable experiences you have had in mathematics?

**Ina:** I studied math in Budapest for a semester - the whole semester was one big memorable experience. I took some very exciting courses, and began realizing that maybe I want to go to grad school and keep studying math.

**Ken:** I always ask this question: do you think there is gender bias in mathematics today?

**Ina:** I keep hearing about it, so maybe there is, but I've never personally experienced anything even remotely close to gender bias. Maybe I've been lucky so far to always be in the right environment. Columbia is the friendliest, most open math department one could imagine. I hope I don't get to experience anything different in the future...

**Ken:** Do you have any hobbies aside from math and ice hockey?

**Ina:** I love drawing, although I haven't done much lately (I was initially going to be a fine arts major in college). Nowadays, I spend most of my free time on ice hockey. I play for the Columbia women's team and at a couple of other places nearby.

*"...don't be afraid to ask questions, communicate with others about what you're learning, and every time you learn something new, try to think of concrete examples."*

**Ken:** Do you have any advice for the girls that come to Girls' Angle?

**Ina:** As I said before, don't be afraid to ask questions. If you're interested in math, ask your teachers to help you find extra reading - there's a lot more to math than what is taught in school, and there are plenty of books out there for young mathematicians that will show you sides of mathematics you didn't even imagine existed.

**Ken:** Thanks, Ina, for this interview! Best of luck on and off the ice!

# A Wine Rack Problem

This problem was described by Ross Honsberger in his book *Mathematical Diamonds*. There, he attributes the observation to Charles Payan of the Laboratoire de Structures Dècretées et Didactique in France. It is the subject of *Anna's Math Journal* on page 10. If you like this problem, check out Ross Honsberger's books, some of which are listed below.

## The Problem

You are stacking wine bottles in a rectangular wine rack (see figure).

At the bottom of the wine rack, there is room for more than three wine bottles, but not four. (All the wine bottles have the same size.)

For the first layer, you place two bottles in the corners and one in between. For the second layer, you place two wine bottles each of which settles and rests upon two of the wine bottles in the first layer.

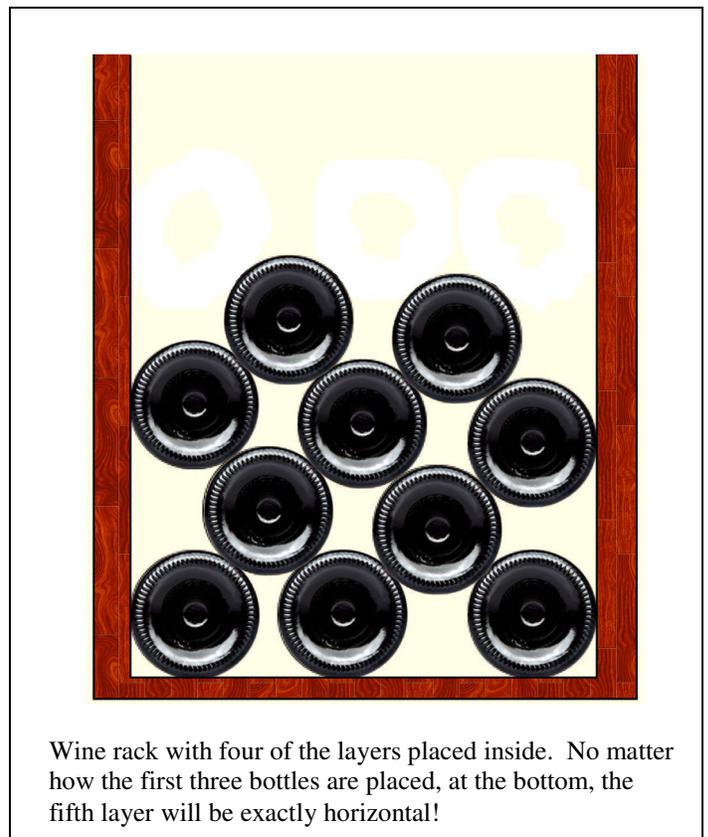
Of course, the two wine bottles in the second layer won't likely be at the same height because that only happens if the middle bottle in the first layer is exactly centered.

Three more wine bottles are placed inside to make a third layer with two of these resting on the sides of the rack.

A fourth layer is added consisting of two more wine bottles. Like the second layer, each of these wine bottles rests upon two wine bottles in the third layer.

So far, only the first of these layers is guaranteed to be perfectly horizontal. The fourth layer could be tilted quite a bit, in fact!

Yet...prove that when you add a fifth layer of three wine bottles, it will be exactly horizontal!



## Other books by Ross Honsberger

Ingenuity in Mathematics  
Mathematical Gems I and II  
Mathematical Delights  
Mathematical Plums  
Mathematical Morsels

# A Divisibility Worksheet

How far can you get with this worksheet? No calculators allowed!

1. Does the number in the first column divide evenly into the number in the second column?

$a$	$b$	$a \mid b?$
2	4	
3	12	
6	16	
5	100	
80	75	

$a$	$b$	$a \mid b?$
7	49	
11	101	
12	60	
12	66	
9	4536	

$a$	$b$	$a \mid b?$
11	14641	
30	43210	
17	493	
29	493	
13	677	

2. What are all the ways to arrange 30 chairs into a rectangular arrangement with no chairs left over? How about 40 chairs? 50 chairs? 64 chairs? 101 chairs? 360 chairs?

## Casting Out 9s

A number is divisible by 9 if and only if the sum of its digits is divisible by 9.

3. Consider the numbers 3, 33, 333, 3333, 33333, etc. Which of these are divisible by 9?

4. Consider numbers that are composed of products of twos and threes. For example, 6, 12, 9 and 54 all only involve the prime factors two or three. Such numbers can be written using exponents as  $2^a 3^b$ . This is the number with  $a$  factors of 2 and  $b$  factors of 3. How many factors does  $2^a 3^b$  have? Can you find a formula in terms of  $a$  and  $b$ ?

5. Suppose  $p$  and  $q$  are two distinct prime numbers. How many factors does  $p^a q^b$  have?

6. How many factors does  $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 19$  have?

7. What is the last digit of  $2^{100}$ ? What are the last two digits of  $2^{100}$ ?

## Fermat's "Little" Theorem

If  $p$  is a prime number, then  $p$  divides  $n^p - n$  for all  $n$ .

The answers to the next 3 problems are all integers.

8. What is the 5<sup>th</sup> root of 248,832?

9. What is the 7<sup>th</sup> root of 27512614111?

10. What is the 13<sup>th</sup> root of 6306757703256894179732154261504?

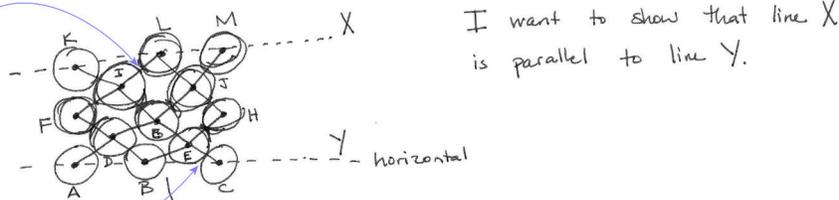
Send in your solutions to [girlsangle@gmail.com](mailto:girlsangle@gmail.com)!

# Anna's Math Journal

By Anna Boatwright

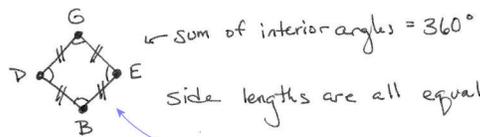
Mathematics is a journey of discovery. As mathematicians take this journey, they follow many wrong turns, believe many incorrect facts, and encounter many mysteries. Out of these twists and turns comes the reward of truth and understanding. However, if you look at math books, you might get the impression that mathematicians rarely err. In this column, Anna Boatwright gives us a peek into her mathematical process of discovery, bravely allowing us to watch even as she stumbles.

Here, Anna tackles the wine rack problem described on page 8.



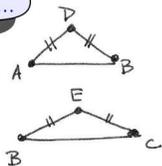
I want to show that line X is parallel to line Y.

The spots where 2 balls touch are important, because I know exactly what the distance is from the center of one ball to the center of another.

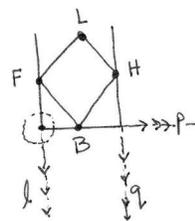


The side lengths of each such parallelogram are fixed at  $2r$ . The angles are unknown so far. The angles might be unequal...

I will start with the bottom of the cellar and work my way up...



These are both isosceles triangles with 2 sides equal to  $2r$ , if "r" stands for the radius of a wine bottle. Note,  $AB \neq BC$ . ← At least, not necessarily.



B is fixed along the horizontal line p.  
H is fixed along the vertical line q.  
F is fixed along the vertical line l.

Forget trying to draw circles, the center point is enough to specify exactly where the bottle is. (And it is much easier to draw!) Imagine a line parallel to the floor that runs through point B, call it q, and imagine 2 lines perpendicular to line p that run through point F and point H, call them l and p, respectively.

Key:

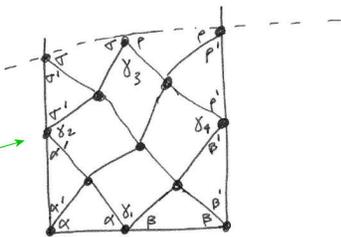
Anna's thoughts

Anna's afterthoughts

Editor's comments

I know that  $\angle A + \angle D + \angle E + \angle C = 180^\circ$

Drawing this diagram out step-by-step helped me a lot. It showed me that by stacking the bottles one by one (after the 1st three are placed), there is exactly one place a bottle can be be, based on where the others have already been laid down. In other words, my choice of where to draw each line and dot is gone, the position is already fixed (after the first three bottles).



FACTS:

$$\alpha + \alpha' = 90^\circ$$

$$\beta + \beta' = 90^\circ$$

$$\rho + \rho' = 90^\circ$$

$$\gamma + \gamma' = 90^\circ$$

$$\alpha + \delta_1 + \beta = 180^\circ$$

$$\alpha' + \delta_2 + \gamma' = 180^\circ$$

$$\beta' + \delta_4 + \rho' = 180^\circ$$

Show that:

$$\gamma' + \gamma = 90^\circ$$

OR

$$\rho' + \rho = 90^\circ$$

OR

$$\gamma + \delta_3 + \rho = 180^\circ$$

Maybe now the problem is reduced to algebra...

Thus,

$$\alpha + \delta_1 + \beta = 180^\circ$$

$$\alpha' + \delta_2 + \gamma' = 180^\circ$$

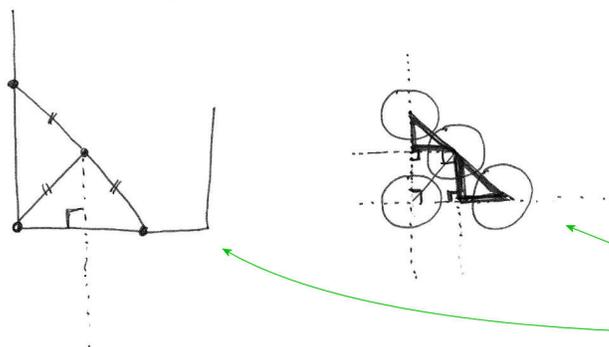
$$+ \beta' + \delta_4 + \rho' = 180^\circ$$


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$$\underbrace{\alpha + \alpha'}_{90^\circ} + \underbrace{\beta + \beta'}_{90^\circ} + \gamma + \rho' + \delta_1 + \delta_2 + \delta_4 = 540^\circ$$

$$\gamma + \rho' + \delta_1 + \delta_2 + \delta_4 = 360^\circ$$

This doesn't quite seem to get me to what I want to show. What else can I think about... what else can I see here...?



Look at these diagrams, do they give you any ideas about a solution...?

Can you help Anna finish this problem?

Key:

Anna's thoughts

Anna's afterthoughts

Editor's comments

ABB 10.23.09

# The Fibonacci Function

by Ken Fan

At the club, this question came up: Is there a formula for the  $n$ th Fibonacci number?

Let  $F_n$  be the  $n$ th Fibonacci number. We can define  $F_n$  by declaring that  $F_1 = F_2 = 1$  and, for integers  $n > 1$ , we have  $F_{n+1} = F_n + F_{n-1}$ . The first few terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, etc.

First, I want to point out that the very definition of the Fibonacci numbers gives us a way to compute  $F_n$  for any  $n$ . We just have to start writing the sequence from the beginning and add consecutive terms to get the next term. Just keep on going until you arrive at the desired Fibonacci number. In a certain sense, there is no difference between having a *formula* and having an explicit set of instructions, or algorithm, for computing the Fibonacci numbers. I'll say more on this point at the end.

However, from the context in which the question was asked at the club, the question was really about whether there is an explicit formula for the  $n$ th Fibonacci number that involves operations such as addition, subtraction, multiplication, division and exponentiation.

There is! And as promised, here's one way to derive that formula.

Looking at the first few terms, the Fibonacci sequence seems to grow pretty fast. In fact, it almost looks like each term is about 60% more than the previous term, i.e. each term is about 1.6 times the prior term. That's not exactly right, but it seems roughly correct. In the table below, the top row shows the first 12 Fibonacci numbers and the bottom row shows ratios of consecutive Fibonacci numbers rounded to the nearest hundredth.

1	1	2	3	5	8	13	21	34	55	89	144
	1.00	2.00	1.50	1.67	1.60	1.63	1.62	1.62	1.62	1.62	1.62

In other words, the Fibonacci numbers are a bit like the sequence  $1, r, r^2, r^3, r^4, r^5$ , etc., where  $r$  is approximately 1.62. Such a sequence, where each term is obtained by multiplying the previous number by a fixed constant is called a **geometric sequence**.

We can't make the sequence  $1, r, r^2, r^3, \dots$  correspond exactly to the Fibonacci sequence. For one thing, that would require the second term, which is  $r$ , to be equal to 1, but if  $r = 1$ , then all the terms would equal 1 because 1 raised to any power is just 1 again.

Well, if a non-constant geometric sequence cannot start off like the first two terms of the Fibonacci sequence, can a geometric sequence at least share the recurrence property of the Fibonacci numbers? In other words, is there a value of  $r$  where each term in the geometric sequence is the sum of the previous two terms? If there were such an  $r$ , it would have to satisfy  $r^{n+2} = r^{n+1} + r^n$ . The solution  $r = 0$  works, but that leads to the uninteresting geometric sequence 1, 0, 0, 0, 0, ... So let's assume  $r$  is not zero and divide both sides of our equation by  $r^n$ . We get the quadratic equation  $r^2 = r + 1$ . The two solutions to this quadratic equation are:

$$a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2}.$$

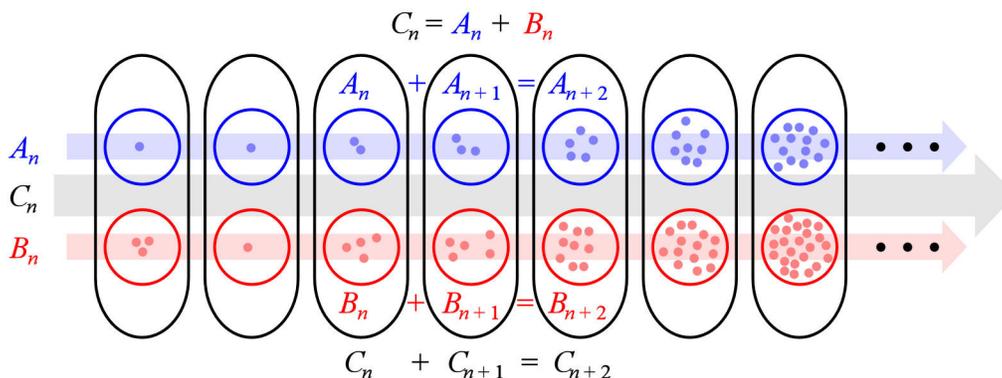
In other words, we have found two interesting geometric sequences that have the property that they start with the number 1 and, if you add consecutive terms, you get the next term. They are  $1, a, a^2, a^3, \dots$  and  $1, b, b^2, b^3, \dots$

Does it surprise you that  $a$  is approximately 1.62? Using a calculator, I find that  $a = 1.61803\dots$

So far, we've found 3 different sequences that start with 1 and satisfy the Fibonacci recurrence property that each term (after the second term) is the sum of the previous two terms. For one of them, the Fibonacci sequence, we're looking for an explicit formula. For the other two, the two geometric sequences, we already have explicit formulas. I'll call any sequence that satisfies the Fibonacci recurrence property a "Fibonacci-like" sequence.

Take a Fibonacci-like sequence and imagine a row of sand piles, where the weight of the  $n$ th sand pile is the  $n$ th term of the sequence. So, for instance, for the Fibonacci numbers, the first pile of sand would weigh 1 pound, the second also 1 pound, the third, 2 pounds, the fourth, 3 pounds, and so on. If I were to combine the sand in two consecutive sand piles, because of the recurrence formula, I would get a sand pile of the exact same weight as the next sand pile.

Now imagine two rows of sand piles lined up side by side each representing different sequences that are both Fibonacci-like. Suppose that the two rows of sand piles were combined to form a single row of sand piles by merging the sand in the  $n$ th sand piles in each row into a single sand pile. Would the resulting sequence of sand piles also be Fibonacci-like? See the illustration below and think about this for a moment.



The sum of two Fibonacci-like sequences is also a Fibonacci-like sequence.

The answer is: sure it would! The weight of two sand piles is just the sum of the weights of the individual sand piles, so everything adds up perfectly. (As an exercise, demonstrate this truth algebraically.)

Multiplying the amount of sand in every sand pile by the same constant also yields a Fibonacci-like sequence (right?).

In other words, because  $1, a, a^2, a^3, \dots$  and  $1, b, b^2, b^3, \dots$  both satisfy the Fibonacci recurrence formula, so would the sequence whose  $n$ th term is given by  $Ma^{n-1} + Nb^{n-1}$ , where  $M$  and  $N$  are constants. Suddenly, we have *infinitely many* Fibonacci-like sequences! If we can find  $M$  and  $N$  so that the first two terms are both one, then not only would that sequence be Fibonacci-like, it *would have to be the one and only Fibonacci sequence!*

That is, if  $M + N = 1$  and  $Ma + Nb = 1$ , then  $Ma^{n-1} + Nb^{n-1}$  would be the  $n$ th Fibonacci number. So let's just solve these two equations for  $M$  and  $N$ ! (It's a system of linear equations: two linear equations with two unknowns.) Try to solve for  $M$  and  $N$  before reading on.

There are many ways to solve the equations  $M + N = 1$  and  $Ma + Nb = 1$  for  $M$  and  $N$ . Katy Bold also solves this kind of system of equations in this issue's *Math in Your World* column (see page 17). In her column she used elimination. Here, I'll illustrate the method of substitution.

From  $M + N = 1$ , I solve for  $M$  to get  $M = 1 - N$ .

I substitute this formula for  $M$  into  $Ma + Nb = 1$  to get a single equation involving only the unknown  $N$ :  $(1 - N)a + Nb = 1$ .

I solve this last equation for  $N$ :

$$\begin{aligned} (1 - N)a + Nb &= 1 \\ a - Na + Nb &= 1 && \text{(distributive law)} \\ N(b - a) &= 1 - a \\ N &= \frac{1 - a}{b - a}. \end{aligned}$$

I substitute this value of  $N$  into  $M = 1 - N$  to find  $M$ :

$$M = 1 - N = 1 - \frac{1 - a}{b - a} = \frac{b - a}{b - a} - \frac{1 - a}{b - a} = \frac{b - 1}{b - a}.$$

We can simplify these formulas by using the fact that  $a + b = 1$ , so that  $1 - a = b$  and  $b - 1 = -a$ :

$$M = \frac{a}{a - b} \text{ and } N = \frac{b}{b - a}.$$

We conclude that  $F_n = Ma^{n-1} + Nb^{n-1} = \frac{a}{a - b} a^{n-1} + \frac{b}{b - a} b^{n-1} = \frac{1}{a - b} (a^n - b^n)$ .

So here's the final formula:

The  $n$ th Fibonacci number is given by  $\frac{a^n - b^n}{a - b}$  where  $a = \frac{1 + \sqrt{5}}{2}$  and  $b = \frac{1 - \sqrt{5}}{2}$ .

Notice that, although this is an explicit formula using only exponentiation, subtraction and division, using this formula to actually compute a specific Fibonacci number can be a rather laborious task. Would you prefer finding the twentieth Fibonacci number by using this formula or the recurrence formula? On the other hand, having such an explicit formula can be useful to help answer other questions. For example, can you use the formula to see why ratios of consecutive Fibonacci numbers do, in fact, get closer and closer to  $a$ ?

One last question: Can you show that every Fibonacci-like sequence is of the form

$$Ma^{n-1} + Nb^{n-1}$$

for some constants  $M$  and  $N$ ? For example, the Lucas numbers are a Fibonacci-like sequence that begin like this: 2, 1, 3, 4, 7, etc. What values of  $M$  and  $N$  correspond to these numbers?

# The Extended Euclidean Algorithm

by Doris Dobi

As promised in the Euclidean Algorithm article in the previous issue, this article discusses the Extended Euclidean Algorithm.

Recall that for any two integers  $a$  and  $b$  we may use division to write

$$\begin{aligned}a &= q_0 \cdot b + r_0 \\b &= q_1 \cdot r_0 + r_1 \\r_0 &= q_2 \cdot r_1 + r_2 \\r_1 &= q_3 \cdot r_2 + r_3 \\&\vdots \\&\vdots \\&\vdots\end{aligned}$$

where the  $q_k$  are all nonnegative integers,  $0 \leq r_0 < b$ , and  $0 \leq r_{k+1} < r_k$ , for  $k \geq 0$ . The process eventually ends when one of the  $r_k$  is zero. Suppose  $r_n$  is the last nonzero  $r_k$ . Then we saw last time that  $r_n$  is the greatest common divisor of  $a$  and  $b$ , which we denoted by  $\text{GCD}(a, b)$ . Now, by backtracking all of our steps we can actually solve for integers  $x$  and  $y$  which satisfy the equation  $ax + by = r_n$ . This process of finding such an  $x$  and  $y$  is commonly referred to as the Extended Euclidean Algorithm. To fix ideas, let  $a = 120$  and  $b = 23$  and consider the problem of finding integers  $x$  and  $y$  such that  $120x + 23y = \text{GCD}(120, 23)$ .

From the Euclidean Algorithm article we can do this by writing:

$$\begin{aligned}120 &= 5 \cdot 23 + 5 \\23 &= 4 \cdot 5 + 3 \\5 &= 1 \cdot 3 + 2 \\3 &= 1 \cdot 2 + 1 \\2 &= 2 \cdot 1 + 0\end{aligned}$$

hence,  $\text{GCD}(120, 23) = 1$ . (By the way, we call two numbers whose greatest common divisor is 1 **relatively prime**). So that by backtracking our work we can write:

$$\begin{aligned}1 &= 3 - 1 \cdot 2 \\&= (23 - 4 \cdot 5) - 1 \cdot (5 - 1 \cdot 3) \\&= 23 - 5 \cdot 5 + 3 \\&= 23 - 5 \cdot (120 - 5 \cdot 23) + (23 - 4 \cdot 5) \\&= 27 \cdot 23 - 5 \cdot 120 - 4 \cdot (120 - 5 \cdot 23) \\&= -9 \cdot 120 + 47 \cdot 23\end{aligned}$$

Hence we find that  $120 \cdot -9 + 23 \cdot 47 = 1$ , and therefore  $x = -9$  and  $y = 47$  is a solution. There are actually infinitely many solutions, but they can all be obtained from this solution by adding  $23t$  to  $x$  and subtracting  $120t$  to  $y$  for any integer  $t$ . Can you see why?

Here's a problem:

You have an unlimited supply of 11 inch sticks and 7 inch sticks. How can you use these sticks to measure off a length of 15 inches?

To solve this problem, we can try to figure out if we can line up the sticks in such a way that the distance between the endpoints of two of the sticks is exactly 15 inches. If we line up  $x_1$  of the 11 inch sticks and  $y_1$  of the 7 inch sticks (end to end), then the ends of the first and last stick will be  $11x_1 + 7y_1$  inches apart. So, let's find integers  $x_1$  and  $y_1$  such that  $11x_1 + 7y_1 = 15$ . Try this yourself before reading on!

Because the greatest common divisor of 7 and 11 is 1 we can use the extended Euclidean algorithm to find integers  $x$  and  $y$  such that  $7x + 11y = 1$ .

$$\begin{aligned} 11 &= 1 \cdot 7 + 4 \\ 7 &= 1 \cdot 4 + 3 \\ 4 &= 1 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0 \end{aligned}$$

Again, we work backward to rewrite 1 in terms of 7 and 11:

$$\begin{aligned} 1 &= 4 - 1 \cdot 3 \\ &= 4 - 1 \cdot (7 - 1 \cdot 4) \\ &= 2 \cdot 4 - 1 \cdot 7 \\ &= 2 \cdot (11 - 1 \cdot 7) - 1 \cdot 7 \\ &= 2 \cdot 11 - 3 \cdot 7 \end{aligned}$$

Hence,  $1 = 2 \cdot 11 - 3 \cdot 7$ . To get a solution to  $11x_1 + 7y_1 = 15$ , we multiply both sides of the equation by 15 and get  $15 = 30 \cdot 11 - 45 \cdot 7$ , so  $x_1 = 30$  and  $y_1 = 45$  is one possible solution. What are other solutions?

It is a good exercise to show that  $\text{GCD}(a, b)$  is the smallest positive integer that can be written as a linear combination of  $a$  and  $b$ .

Suppose we have  $x_0$  and  $y_0$  that satisfy  $ax_0 + by_0 = \text{GCD}(a, b)$ . Can you show that all solutions to  $ax + by = \text{GCD}(a, b)$  are given by:

$$\left(x_0 + \frac{kb}{\text{GCD}(a,b)}, y_0 - \frac{ka}{\text{GCD}(a,b)}\right)$$

where  $k$  is an integer?

Also, given integers  $a$  and  $b$ , for what values of  $n$  are there integer solutions in  $x$  and  $y$  to the equation  $ax + by = n$ ?

# Cooking With Equations

By Katy Bold

After hearing about LäraBars from several friends, I decided to try one. LäraBars are similar to granola bars but they only have a few ingredients, such as dates, nuts, raisins, coconut, cocoa, salt, and spices. Based on recommendations, I tried a “Peanut Butter Cookie” LäraBar, and it was great. The only problem was the price. Even buying the bars in bulk, they cost about \$1.50 each.

The Peanut Butter Cookie LäraBar only has three ingredients: dates, peanuts, salt. Because the ingredient list is so simple, I decided to try making my own “KatyBar” at home. I chopped up dates and put them in a food processor with some peanuts (a blender would also work well for this). When the dates and peanuts were chopped up into tiny pieces, I formed the mixture into small bars.

My first batch was okay. Not as good as a true LäraBar, but not bad either. I used about 66% dates and 33% peanuts, and it seemed like the “KatyBar” needed more peanuts. But how much more?

There were two options for how to proceed:

1. Keep making “KatyBars” until I liked the result. Along the way, I would need to keep track of the proportion of ingredients in each batch.
2. Use math to determine a good mixture of dates and peanuts.

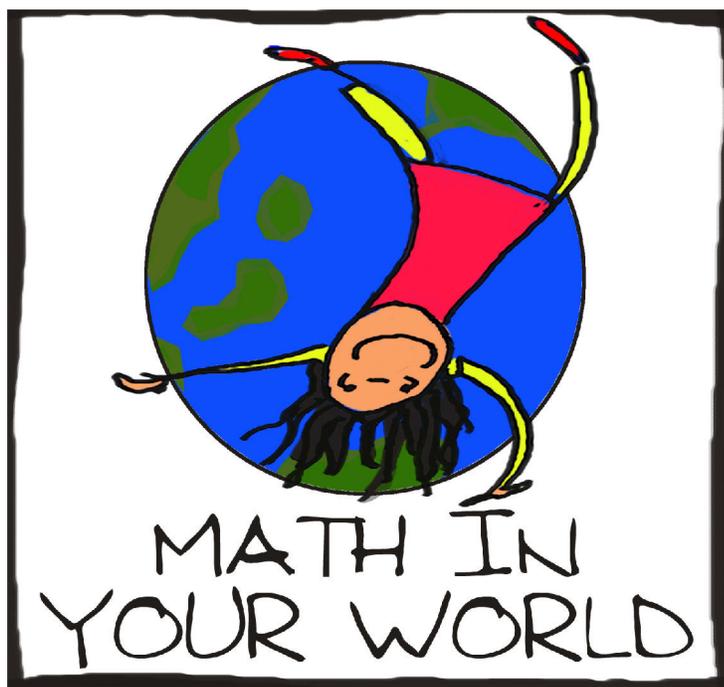
Well, of course I took the second option! Let’s turn this cooking problem into a math problem.

I want to mix peanuts and dates together, but I do not know how much of each to include. The quantity of each ingredient is a variable:

$$\begin{aligned} P &= \text{units of peanuts (hectograms)} \\ D &= \text{units of dates (hectograms)} \\ S &= \text{salt}^* \end{aligned}$$

\*I am less concerned about getting the right amount of salt than I am about getting the right amount of peanuts and dates.

If we ignore the salt, that gives two unknowns. To solve for two unknowns, we generally need two equations, which can come from nutritional information: Mass and Nutritional Content (Calories, fat, etc.).



Logo Design by Hana Kitasei



Following is nutritional information about the Peanut Butter Cookie LäraBar and its ingredients:

	Mass (g)	Calories (Cal)	Sugar (g)	Fat (g)	Protein (g)
<b>LäraBar</b>	48	210	16	13	5
<b>Dates (Deglet Noor)</b>	100	282	63.35	0.39	2.45
<b>Peanuts</b>	100	567	3.97	49.24	25.8

Let's start with the mass and calorie equations.

Mass (g)	(Mass of dates) $\times$ (Units of dates) + (Mass of peanut) $\times$ (Units of peanuts) = (Mass of LäraBar) $100D + 100P = 48$
Calories (Cal)	(Cal. of dates) $\times$ (Units of dates) + (Cal. of peanuts) $\times$ (Units of peanuts) = (Cal. of LäraBar) $282D + 567P = 210$

Can you construct the equations for sugar, fat, and protein?

Sugar (g)	
Fat (g)	
Protein (g)	

Each of these equations is **linear** because the exponents of the variables  $D$  and  $P$  are all 1. You probably are also familiar with **nonlinear** equations. The simplest nonlinear equations are quadratic:  $ax^2 + bx + c = 0$ . In general, it is easier to solve linear equations than nonlinear equations. Since we have more than one linear equation, we have a **system of linear equations**. There are several ways you can solve a system of linear equations, but they all have the same feature: somehow remove the variables until you are left with a single equation with only a single variable.

Let's use the Mass and Calorie equations to solve for  $D$  and  $P$ .

Try solving this yourself before turning the page!



As mentioned, there are many ways to solve for  $D$  and  $P$ . The method we'll use here is called "elimination".

Mass (g):	$100D + 100P = 48$
Calories (Cal):	$282D + 567P = 210$
Multiply the Mass equation by -2.82:	$-282D - 282P = -135.36$
Add the last two equations to eliminate $D$ :	$285P = 74.64$
Solve for $P$ :	$P = \frac{74.64}{285}$ $\approx 0.26189$
Solve for $D$	$100D + 100P = 48$ $100D + 26.189 \approx 48$ $D \approx 0.21811$

The ratio of Peanuts to Dates is about 0.26189 : 0.21811, or roughly 5 : 4. So, for every 4 units of dates that I use, I need to add 5 units of peanuts. That's why my "KatyBars" needed more peanuts! I was using twice as much dates as peanuts.

### Take It To Your World

If we tried to make a bar with 5 ingredients – how many equations are needed so that all amounts would be specified? What are some possible sources for the equations?

If you check the amount of protein in a "KatyBar," you'll notice that it's not the same as in a LäraBar. Why might that be?

Try making your own version of a LäraBar at home!

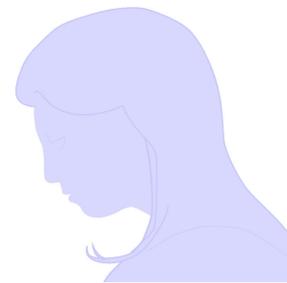
Think about a snack or food you like. Can you figure out the ratio of its ingredients?

Sources: *LäraBar.com* and the *United States Department of Agriculture (USDA)*

# The Adventures of Emmy Newton

## Episode 1. *Mr. Wheel's Mysterious Sequence*

by Maria Monks



Emmy Newton was the brightest student in her fifth grade class. She didn't really try to be this way; she wasn't the kind of student who obsessed over grades. She just really loved to learn, and paid full attention in every class. She especially loved math, and started carrying around a pencil and paper with her wherever she went.

After all, her math teacher, Mr. Wheel, was awesome. Every morning, he would come running into the classroom, excitedly waving his arms, and exclaim, "Guess what?"

The loud chatter in the classroom would come to an abrupt halt.

"What, Mr. Wheel?" the class would say in unison. Only Priscilla and Amy, the popular girls, remained silent. Oh well, Emmy thought, they were the ones missing out on the excitement.

"I just realized something!" he would say. Then he would grab the big yellow chalk and start writing something mathematical and mysterious on the board.

Today, he started writing numbers in a sequence:

*0, 1, 1, 2, 3, 5, 8, 13, 21, ...*

"Shall I keep going?" asked Mr. Wheel.

"Wait, stop," said Melissa Leibnitz, who wanted to try to figure out the pattern in the numbers before he continued. Melissa was Emmy's best friend. She was very excited about math as well, although she didn't understand things quite as quickly as Emmy. Melissa kind of admired Emmy, and really liked when her friend would explain things to her. It was always rather magical.

"No, don't stop," said Emmy, who already figured out the pattern. She smiled mischievously. "You have to keep writing those numbers forever."

Melissa burst into a fit of giggles. Priscilla snorted, as if she couldn't believe Emmy was so dumb. Mr. Wheel smiled. "I would if there were an infinite supply of chalk in the world. Alas, there is a finite amount of matter on this planet... But go on, Emmy, why do you think I should keep going on forever?"

Emmy replied, "The sequence never ends. It starts with the numbers 0 and 1, and every number after that is the sum of the previous two numbers." She had read about this sequence in the problem book that Mr. Wheel gave her for winning the Factorization Bee. "It's called... something like... I think it's the Fibonacci sequence."

"Precisely!" exclaimed Mr. Wheel. He started hopping up and down on his left foot. Melissa was amazed. She started adding up the numbers in her head, checking that it worked. Yes, if you add 0 and 1 you get 1, then you add 1 and 1 and you get 2, then  $1 + 2 = 3$ ... it keeps going, and 8 plus 13 is 21. So the next number should be... let's see... 13 plus 21...

"So what should the next number be?" asked Mr. Wheel.

"34!" Emmy and Melissa called out in unison. "34, yeah," said a few others, as they caught on to the pattern.

"Then 55," said Kyle, "and then 89... hey, you should keep writing forever." Priscilla snorted, but no one paid her any attention.

“Precisely!” exclaimed Mr. Wheel. He wrote more numbers on the chalkboard, which now read:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .

“That’s enough for now. What else do you notice about these numbers?”

Emmy didn’t really know much about the Fibonacci numbers besides how to find the next number in the sequence. But she knew that looking at even and oddness, called “parity,” is often useful when studying whole numbers like these.

She picked up her pencil and started writing whether each number was even or odd, in a pattern. Her paper looked like this:

*Even, Odd, Odd, Even, Odd, Odd, Even, Odd, Odd, Even, Odd, Odd, Even*

Fascinating! Maybe this is what Mr. Wheel realized this morning. Emmy raised her hand. Mr. Wheel pivoted on his left foot, which he was still standing on, planted his right foot down on the ground, and pointed at Emmy. “Yes, Miss Newton?”

“Every third number in the sequence is even, and the rest are odd.”

Melissa looked up at the board in astonishment, and started whispering to herself, “Zero is even, one is odd, one is odd, two is even, ...”

“It certainly appears so,” said Mr. Wheel, “And why would that be? Would it keep up that pattern if I had that infinite supply of chalk we’ve all been longing for?”

“Yes,” said Emmy. Thinking quickly, she came up with a reason. “If you add an odd number to an even number you get an odd number, and if you add two odd numbers, you get an even number. So whenever there is an even followed by an odd, the next number will be odd. Then you add these two odds together, and you get an even again. Finally, you add the second odd and the new even, and you get an odd, and the process starts all over again.”

“Huh?” said David and Melissa in unison. There were too many evens and odds in that sentence.

Mr. Wheel scratched his head with the chalk, making a big yellow splotch on his unkempt black hair. “You’re right, but let’s do it carefully. Let’s see if we can always predict whether the next number will be even or odd.”

He wrote “Even, Odd” on the chalkboard. “The sequence starts with an even number followed by an odd number. Let’s say we forgot what the Fibonacci numbers were...” he erased the rest of the board, and went on, “and just knew that the next number is the sum of the two previous numbers. Would the next number be even or odd?”

“Oh, it would be odd, since an even number plus an odd number is odd. I see,” said Kyle.

“Precisely!” exclaimed Mr. Wheel. “And the next number?”

“Even, because an odd plus an odd is an even,” said Melissa. “And then the next number is odd, and yes, it does start over again!”

“That’s right,” said Mr. Wheel. “The pattern repeats from here.” He sat down and started spinning on his wheely chair. After a few moments, he jumped back up again. “Ok, now let me show you what I realized this morning.”



Emmy perked up. Was he going to show them something even more amazing than what they just discovered? Mr. Wheel wrote on the board:

$$\begin{array}{r} 0 \cdot 0 + 1 \cdot 1 = 1 \\ 1 \cdot 1 + 1 \cdot 1 = 2 \\ 1 \cdot 1 + 2 \cdot 2 = 5 \\ 2 \cdot 2 + 3 \cdot 3 = 13 \\ 3 \cdot 3 + 5 \cdot 5 = 34 \\ 5 \cdot 5 + 8 \cdot 8 = 89 \end{array}$$

“Whoa!” said Emmy. “The numbers on the left are always Fibonacci numbers that are right next to each other, and the numbers down the right column are every other Fibonacci number!”

The class erupted in excitement. Mr. Wheel was always magical.

Emmy and Melissa started talking about why this might be true. Meanwhile, the phone rang by the door in the classroom.

“Hello... Uh huh,” said Mr. Wheel into the phone, “Yes, certainly. Emmy and Melissa,” he called, “Will you two go down to the principal’s office? He wants to send me something.”

The two girls nodded, and headed downstairs, still talking about the Fibonacci numbers.

“So does that pattern continue?” wondered Melissa aloud.

“Let’s see. The next equation, following that pattern, should be  $8 \cdot 8 + 13 \cdot 13$ . That’s  $64 + 169$ , which is 235. Is 235 the second Fibonacci number after 89?”

Melissa liked computing more Fibonacci numbers. “I remember 144 comes after 89, and so the next number is  $144 + 89$ , which is 235! It works! Let’s do it again!”

“No,” said Emmy, “Let’s try and think about why it’s true...” her mind was drifting off, and she didn’t even realize they had gone down one too many flights of stairs.

Melissa suddenly realized she didn’t know where she was. “Emmy, don’t you think we went down too far? I think we’re in the basement,” she whispered nervously.

“Hmm, you might be right,” said Emmy, looking around. She kept walking down the stairs, and came to a rusty-looking door. She was too curious to turn back. “I’ve never been in the basement. Maybe there’s a way under the school to get to the principal’s office!”

“But maybe we’re not allowed...” said Melissa.

“Don’t be silly, of course we’re allowed. Mr. Wheel told us to go to the Principal’s office, but he didn’t tell us how we had to get there. I bet this door leads through to the other side of the building, where we can take the other stairwell up to the first floor.” Emmy had a good sense of direction, and she loved maps. She once studied a map of her school. It was shaped like a square, with a smaller square forming the border of the courtyard in the middle. There were classrooms and offices all along the sides of the inner and outer squares on each of the first, second, and third floors.

But the map never showed the basement! Emmy was very excited to see what it looked like below the school.

She pulled on the handle of the door. It wouldn’t budge. She pulled harder and realized it was just a heavy door. It opened creakily, and the two girls peered into the basement corridor.

Emmy and Melissa gasped in astonishment at the sight that lay ahead.

*TO BE CONTINUED...*



# Notes from the Club

These notes cover some of what happened at Girls' Angle meets. They are far from being complete. In these notes, we include some of the things that you can try or think about at home or with friends. We also include some highlights and some elaborations on meet material.

## Session 5 – Meet 1 – September 10, 2009

Mentors: Lauren McGough, Jennifer Melot, Mia Minnes, Maria Monks

The fifth session of Girls' Angle began with tricky introductions. As an ice breaker activity, girls were asked to tell us about themselves. However, each girl was given a special restriction that they had to obey. For example, one girl was only allowed to use prime numbers. Another could only use perfect squares. Another had to speak in sentences that were exactly 4 words long. Despite these restrictions, the girls managed to say quite a bit about themselves as they concocted all manner of schemes to express themselves subject to the restrictions. How would you answer the question, "How old are you?" or "In what year were you born?" if the only numbers you were allowed to use were perfect squares?

## Session 5 – Meet 2 – September 17, 2009

Mentors: Eli Grigsby, Jennifer Melot, Maria Monks

Special Visitor: Tanya Khovanova, mathematician

Tanya performed some magic tricks that are based on properties of the binary representation system. For example, many people are familiar with the game of guessing a secret number between 1 and 100 by asking no more than seven "yes/no" questions. Most people proceed by asking a question such as, "is the number higher than 50?", and then, based on the answer, formulate their next question. If the number is higher than 50, then the next question typically is, "Is it higher than 75?", otherwise they might ask, "Is it higher than 25?"

However, Tanya challenged the girls to come up with a way to play the guessing game by *writing down all seven questions in advance*. The trick is to make the  $n$ th question be: "Is the binary digit of the number in the  $2^{n-1}$  place equal to zero?"

After Tanya's visit, some girls worked with Eli on a divisibility worksheet (see page 9) while others played a game introduced at the first meet called "Special Ops". In Special Ops, one girl, the special ops girl, plays the role of a function. The other girls try to guess what this function is by feeding the special ops girl inputs to her function. The special ops girl applies her function to the input and states the output. Maria and Jennifer moderated.

Two of the girls came up with a curious function whose first few values are shown in the table below. Can you think of a function that is consistent with this table?

$n$	1	2	3	4	5	6	7	8	9	10
$f(n)$	3	3	5	4	4	3	5	5	4	3

Session 5 – Meet 3 – September 24, 2009

Mentors: Kristina Cibor, Lauren Cipicchio, Jennifer Melot, Maria Monks, Julia Yu

The question arose as to whether or not there is an explicit (non-recursive) formula for the  $n$ th Fibonacci number. Please see page 12.

Session 5 – Meet 4 – October 1, 2009

Mentors: Lauren Cipicchio, Ariana Mann, Julia Yu

Lauren and Julia worked with the girls on logic while Ariana worked with girls on orders of numbers. If you like logic puzzles, you might enjoy reading books by Raymond Smullyan.

Session 5 – Meet 5 – October 8, 2009

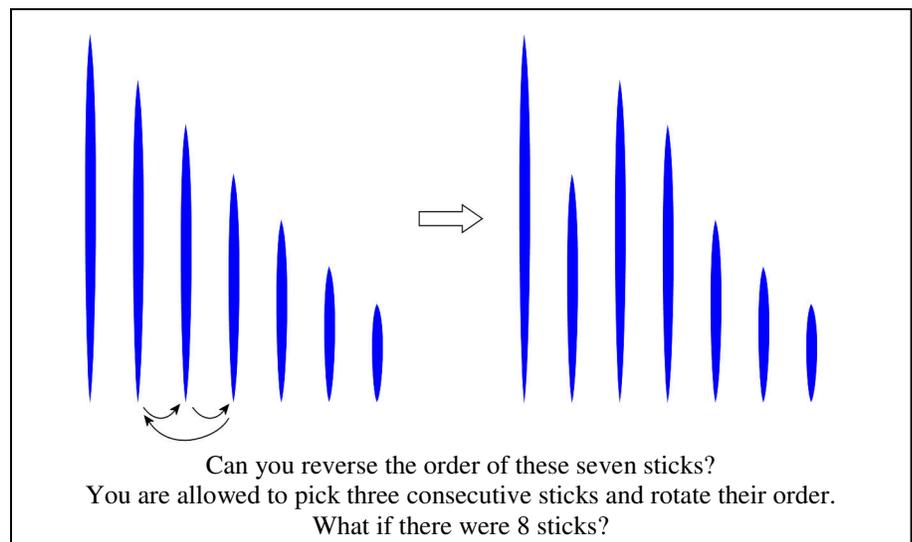
Mentors: Lauren Cipicchio, Lauren McGough, Maria Monks, Katherine Paur, Julia Yu

Special Visitor: Katherine Paur, Kiva Systems

Katherine talked about how to get lots of robots to get from one place to another in the most efficient way possible. She gave the girls an exercise on an 8 by 8 grid. Each girl represented a robot and stood at an initial position in the grid. Each was given a final destination which they had to reach by moving in single steps along verticals or horizontals. Of course, they had to do this while avoiding any collisions, so whenever two girls wanted to step into the same grid square, they would have to figure out who would get to go there and who would have to stay put. Katherine asked, “How can this be done efficiently?” and “What algorithm can be given to all the robots to solve this task in an efficient way?” She mentioned that the problem of determining the most efficient solution is an NP-hard problem, something we may discuss in the future.

After tackling two such problems, Katherine showed us a computer simulation that showed robots getting trapped in traffic jams and discussed ideas she came up with to eliminate these traffic jams.

Later in the meet, some girls explored permutations and tackled problems like the one shown at right.



Session 5 – Meet 6 – October 15, 2009

Mentors: Lauren Cipicchio, Lauren McGough, Charmaine Sia, Julia Yu

In one of this meet's activities, the girls had to place fifteen numbers in order from least to greatest (without the aid of a calculator). The task sounds easy, but it isn't! By the end of the meet, the numbers were all placed in the correct order, but it was not yet *proven* that the order was, in fact, correct. For example, one question that arose was whether  $2^{330}$  was greater than or less than  $10^{100}$ . After a bit of work, it was correctly guessed that  $2^{330} < 10^{100}$ . We'll present a rigorous proof of this fact here. Some day, I hope members can provide a rigorous proof that the order they came up with is correct.

If we take tenth roots of both sides, we see that we have to show that  $2^{33} < 10^{10}$ .

Compute that  $2^5 = 32$ , so  $2^{10} = 32^2 = 1,024$  and  $2^{11} = 2,048$ . This means  $2^{33} = 2,048^3$ .

We can use the binomial theorem (or just check that  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ ):

$$\begin{aligned} 2,048^3 &= (2,000 + 48)^3 \\ &= 2,000^3 + 3(2,000^2)(48) + 3(2,000)(48^2) + 48^3 \\ &< 2,000^3 + 4(2,000^2)(50) + 4(2,000)(50^2) + 100^3 \\ &= 2,000^3 + 200(2,000^2) + 10,000(2,000) + 1,000,000 \\ &= 8,000,000,000 + 800,000,000 + 20,000,000 + 1,000,000 \\ &= 8,821,000,000 \end{aligned}$$

and this last number is less than  $10^{10}$ , so we're done! (To go from the 2<sup>nd</sup> to the 3<sup>rd</sup> line, I picked larger numbers, which, when substituted, would be easy to compute with without a calculator.)

Also of note, Lauren Cipicchio created a number of algebra secret code puzzles which proved quite popular with the girls.

Session 5 – Meet 7 – October 22, 2009

Mentors: Lauren Cipicchio, Wei Ho, Lauren McGough, Maria Monks, Charmaine Sia, Rediet Tasfaye, Julia Yu

Special Visitor: Jane Kostick, woodworker

Jane presented a number of wooden puzzles that are based on the Platonic solids. By exploiting the high symmetry of these solids, Jane was able to make numerous shapes out of copies of a simple unit, such as a dowel or carefully fashioned beam. (See the cover.)

Julia brought in a classic cake cutting problem which several girls solved. See page 26 for an account of the solution discovered by **Fern** and **cat in the hat**.



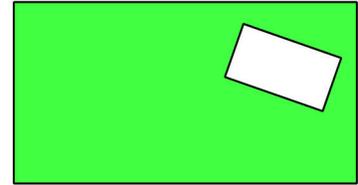
Some of the objects Jane shared with us.

# Member's Thoughts

## Cake Cutting Problem Proof

by Fern

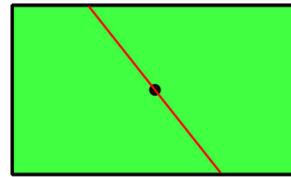
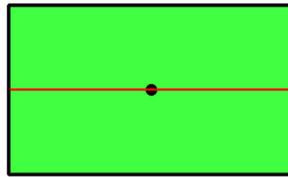
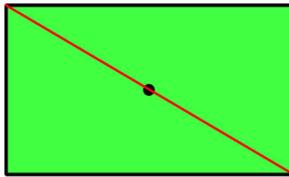
**Problem:** *There is a rectangular cake with one rectangular piece cut out of it. How do you cut the cake into two equal pieces with one straight cut?*



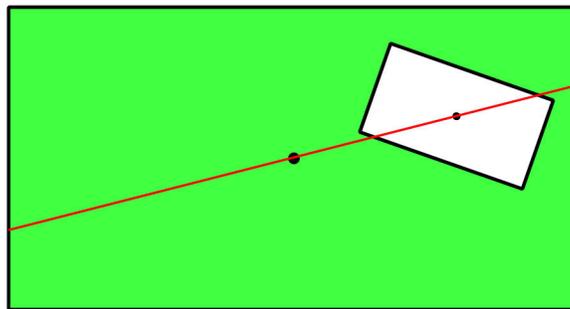
Here's the solution that **cat in the hat** and I found at the club.

The solution of the problem is that you would make a straight cut through both the center of the missing part and the center of the cake when it was whole.

Whenever there is a cut through the center of a rectangle, there are two equal pieces left. For example, a cut from one corner to another leaves two equal pieces, and the cut passes through the center. A cut that is parallel to one of the sides is only cutting the cake in half if it passes through the center. Any cut passing through the center of a rectangle will make two equal (or **congruent**) parts.



Cuts through the center of rectangle divide it into two equal parts.



So, we want to cut the remaining cake into two equal parts. To make this happen, we can try to find a cut that evenly splits the missing part and also evenly splits the original size of the cake. This can be achieved by making a straight line through the center of the original cake that also goes through the center of the missing part.

Each person would “have” half the original cake and half the missing part, so each person would have an equal piece.

# Calendar

## Session 5: (all dates in 2009)

September	10	Start of fifth session!
	17	Tanya Khovanova, mathematician
	24	
October	1	
	8	Katherine Paur, Kiva Systems
	15	
	22	Jane Kostick, wood worker
	29	No meet
November	5	
	12	JJ Gonson, Cuisine En Locale
	19	Allie Anderson, MIT Aeronautics and Astronautics
	26	Thanksgiving - No meet
December	3	Meg Aycinena Lippow, MIT CSAIL
	10	

## Session 6: (all dates in 2010)

January	28	Start of sixth session!
February	4	
	11	Meike Akveld, ETH, Switzerland
	18	No meet
	25	
March	4	
	11	
	18	
	25	No meet
April	1	
	8	
	15	
	22	No meet
	29	
May	6	

# Girls' Angle: A Math Club for Girls

**Gain confidence in math! Discover how interesting and exciting math can be! Make new friends!**

**What is Girls' Angle?** Girls' Angle is a math club for girls and a supportive community for all girls and women engaged in the study, use and creation of mathematics. Our primary mission is to foster and nurture girls' interest and ability in mathematics and empower them to be able to tackle any field, no matter the level of mathematical sophistication required. We offer a comprehensive approach to math education and use a four component strategy to achieve our mission: Girls' Angle mentors, the Girls' Angle Support Network, the Girls' Angle Bulletin and Community Outreach.

**Who are the Girls' Angle mentors?** Our mentors possess a deep understanding of mathematics and enjoy explaining math to others. The mentors get to know each member as an individual and design custom tailored projects and activities designed to help the member improve at mathematics and develop her thinking abilities. Because we believe learning follows naturally when there is motivation, our mentors work hard to motivate. In order for members to see math as a living, creative subject, at least one mentor is present at every meet who has proven and published original theorems.

**What is the Girls' Angle Support Network?** The Support Network consists of professional women who use math in their work and are eager to show the members how and for what they use math. Each member of the Support Network serves as a role model for the members. Together, they demonstrate that many women today use math to make interesting and important contributions to society. They write articles for the Bulletin, take part in interviews and visit the club.

**What is the Girls' Angle Bulletin?** The Girls' Angle Bulletin is a bimonthly (6 issues per year) publication that features interviews, articles and information of mathematical interest. The electronic version is free. The printed version (beginning with volume 3, number 1) is free for members and can be purchased by others at cost. Please contact us if you'd like to purchase printed issues.

**What is Community Outreach?** Girls' Angle accepts commissions to solve math problems from members of the community. Our members solve them. We believe that when our members' efforts are actually used in real life, the motivation to learn math increases.

**Who can join?** Ultimately, we hope to open membership to all women. Currently, we are open primarily to girls in grades 5-11. We aim to overcome math anxiety and build solid foundations, so we welcome *all girls* regardless of perceived mathematical ability. There is no entrance test.

**In what ways can a girl participate?** There are 2 ways: **membership** and **active subscription** to the Girls' Angle Bulletin. **Membership** is granted per session and includes access to the club and extends the member's subscription to the Girls' Angle Bulletin to one year from the start of the current or upcoming session. You can also pay per session. If you pay per session and come for three meets, you will get a subscription to the Bulletin. **Active subscriptions** to the Girls' Angle Bulletin allow the subscriber to ask and receive answers to math questions through email. Please note that we will not answer email questions if we think that we are doing the asker's homework! We currently operate in 12 meet sessions, but girls are welcome to join at any time. The program is individually focused so the concept of "catching up with the group" doesn't apply. Note that you can receive the Girls' Angle Bulletin free of charge. Just send us email with your request.

**Where is Girls' Angle located?** Girls' Angle is located about 10 minutes walk from Central Square on Magazine Street in Cambridge, Massachusetts. For security reasons, only members and their parents/guardian will be given the exact location of the club and its phone number.

**When are the club hours?** Girls' Angle meets Thursdays from 3:45 to 5:45. For calendar details, please visit our website at [www.girlsangle.org](http://www.girlsangle.org) or send us email.

**Can you describe what the activities at the club will be like?** Girls' Angle activities are tailored to each girl's specific needs. We assess where each girl is mathematically and then design and fashion strategies that will help her develop her mathematical abilities. Everybody learns math differently and what works best for one individual may not work for another. At Girls' Angle, we are very sensitive to individual differences. If you would like to understand this process in more detail, please email us!

**Are donations to Girls' Angle tax deductible?** Yes, Girls' Angle is a 501(c)(3). Please make donations out to **Girls' Angle** and send to Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038.

**Who is the Girls' Angle director?** Ken Fan is the director and founder of Girls' Angle. He has a Ph.D. in mathematics from MIT and was an assistant professor of mathematics at Harvard, a member at the Institute for Advanced Study and a National Science Foundation postdoctoral fellow. In addition, he has designed and taught math enrichment classes at Boston's Museum of Science and worked in the mathematics educational publishing industry. Ken has volunteered for Science Club for Girls and worked with girls to build large modular origami projects that were displayed at Boston Children's Museum. These experiences have motivated him to create Girls' Angle.

**Who advises the director to ensure that Girls' Angle realizes its goal of helping girls develop their mathematical interests and abilities?** Girls' Angle has a stellar Board of Advisors. They are:

Connie Chow, executive director of Science Club for Girls  
Yaim Cooper, graduate student in mathematics, Princeton  
Julia Elisenda Grigsby, assistant professor of mathematics, Boston College  
Kay Kirkpatrick, Courant Instructor/PIRE fellow, NYU  
Grace Lyo, Moore Instructor, MIT  
Lauren McGough, MIT '12  
Mia Minnes, Moore Instructor, MIT  
Beth O'Sullivan, co-founder of Science Club for Girls.  
Elissa Ozanne, Senior Research Scientist, Harvard Medical School.  
Kathy Paur, Kiva Systems  
Gigliola Staffilani, professor of mathematics, MIT  
Katrin Wehrheim, associate professor of mathematics, MIT  
Lauren Williams, assistant professor of mathematics, UC Berkeley

**At Girls' Angle, mentors will be selected for their depth of understanding of mathematics as well as their desire to help others learn math. But does it really matter that girls be instructed by people with such a high level understanding of mathematics?** We believe YES, absolutely! One goal of Girls' Angle is to empower girls to be able to tackle *any* field regardless of the level of mathematics required, including fields that involve original research. Over the centuries, the mathematical universe has grown enormously. Without guidance from people who understand a lot of math, the risk is that a student will acquire a very shallow and limited view of mathematics and the importance of various topics will be improperly appreciated. Also, people who have proven original theorems understand what it is like to work on questions for which there is no known answer and for which there might not even be an answer. Much of school mathematics (all the way through college) revolves around math questions with known answers, and most teachers have structured their teaching, whether consciously or not, with the knowledge of the answer in mind. At Girls' Angle, girls will learn strategies and techniques that apply even when no answer is known.

Also, math should not be perceived as the stuff that is done in math class. Instead, math lives and thrives today and can be found all around us. Girls' Angle mentors can show girls how math is relevant to their daily lives and how this math can lead to abstract structures of enormous interest and beauty.

# Girls' Angle: A Math Club for Girls

## Membership Application

Applicant's Name: (last) \_\_\_\_\_ (first) \_\_\_\_\_

Applying For:  Membership (Access to club, premium subscription)  
 Active Subscription (interact with mentors through email)

Parents/Guardians: \_\_\_\_\_

Address: \_\_\_\_\_ Zip Code: \_\_\_\_\_

Home Phone: \_\_\_\_\_ Cell Phone: \_\_\_\_\_ Email: \_\_\_\_\_

**Emergency contact name and number:** \_\_\_\_\_

**Pick Up Info:** For safety reasons, only the following people will be allowed to pick up your daughter. They will have to sign her out. Names: \_\_\_\_\_

**Medical Information:** Are there any medical issues or conditions, such as allergies, that you'd like us to know about? \_\_\_\_\_

**Photography Release:** Occasionally, photos and videos are taken to document and publicize our program in all media forms. We will not print or use your daughter's name in any way. Do we have permission to use your daughter's image for these purposes? **Yes** **No**

**Eligibility:** For now, girls who are roughly in grades 5-11 are welcome. Although we will work hard to include every girl no matter her needs and to communicate with you any issues that may arise, Girls' Angle has the discretion to dismiss any girl whose actions are disruptive to club activities.

**Permission:** I give my daughter permission to participate in Girls' Angle. I have read and understand everything on this registration form and the attached information sheets.

\_\_\_\_\_  
(Parent/Guardian Signature) Date: \_\_\_\_\_

Membership-Applicant Signature: \_\_\_\_\_

- Enclosed is a check for (indicate one) (prorate as necessary)
  - \$216 for a 12 session membership  \$50 for a one year active subscription
  - I am making a tax free charitable donation.
  
- I will pay on a per session basis at \$20/session. (Note: You still must return this form.)

Please make check payable to: **Girls' Angle**. Mail to: Girls' Angle, P.O. Box 410038, Cambridge, MA 02141-0038. Please notify us of your application by sending email to [girlsangle@gmail.com](mailto:girlsangle@gmail.com). Paying on a per session basis comes with a one year subscription to the Bulletin, but not the math question email service. Also, please sign and return the Liability Waiver.

**Girls' Angle: A Math Club for Girls  
Liability Waiver**

I, the undersigned parent or guardian of the following minor(s)

\_\_\_\_\_

do hereby consent to my child(ren)'s participation in Girls' Angle and do forever and irrevocably release Girls' Angle and its directors, officers, employees, agents, and volunteers (collectively the "Releasees") from any and all liability, and waive any and all claims, for injury, loss or damage, including attorney's fees, in any way connected with or arising out of my child(ren)'s participation in Girls' Angle, whether or not caused by my child(ren)'s negligence or by any act or omission of Girls' Angle or any of the Releasees. I forever release, acquit, discharge and covenant to hold harmless the Releasees from any and all causes of action and claims on account of, or in any way growing out of, directly or indirectly, my minor child(ren)'s participation in Girls' Angle, including all foreseeable and unforeseeable personal injuries or property damage, further including all claims or rights of action for damages which my minor child(ren) may acquire, either before or after he or she has reached his or her majority, resulting from or connected with his or her participation in Girls' Angle. I agree to indemnify and to hold harmless the Releasees from all claims (in other words, to reimburse the Releasees and to be responsible) for liability, injury, loss, damage or expense, including attorneys' fees (including the cost of defending any claim my child might make, or that might be made on my child(ren)'s behalf, that is released or waived by this paragraph), in any way connected with or arising out of my child(ren)'s participation in the Program.

Signature of applicant/parent: \_\_\_\_\_ Date: \_\_\_\_\_

Print name of applicant/parent: \_\_\_\_\_

Print name(s) of child(ren) in program: \_\_\_\_\_

